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NOTES AND COMMENTS

A NOTE ON PARTITIONED DETERMINANTS

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A formula for the determinant of a partitioned matrix, possibly with singular submatrices, is derived and applied to some psychometric and numerical problems.

Key words: determinants, eigenvalues, Procrustus rotation, canonical analysis.

Introduction

Suppose

$$M = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \tag{1}$$

where A is $m \times m$, B is $m \times n$, C is $n \times m$, and D is $n \times n$. If A is nonsingular then Schur [1917] has proved that

$$\det(M) = \det(A)\det(D - CA^{-1}B). \tag{2}$$

In this note we derive more general results which can be used if A is singular. From the review of Ouellette [Note 1] it appears that these results are new. In our last section we briefly indicate some possible applications in psychometrics, more specificially in generalizations of canonical analysis.

First Reduction

We distinguish a number of different cases. In case I the matrix A is nonsingular, and formula (2) applies. In case II the matrix A is singular, of rank $m - r$, say. The singular value decomposition of A is

$$A = K_1 \Omega L_1', \tag{3}$$

with K_1 and L_1 columnwise orthonormal $m \times (m - r)$ matrices, and with Ω diagonal with nonzero diagonal entries. Also define the columnwise orthonormal $m \times r$ matrices K_0 and L_0, satisfying $K_0' K_1 = L_0' L_1 = 0$. We now use

$$\det(M) = \det \begin{vmatrix} K_1' & 0 \\ K_0' & 0 \\ 0 & I \end{vmatrix} \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} L_1 & L_0 & 0 \\ 0 & 0 & I \end{vmatrix} = \det \begin{vmatrix} \Omega & 0 & K_1'B \\ 0 & 0 & K_0'B \\ CL_1 & CL_0 & D \end{vmatrix}, \tag{4}$$

where we assume that the signs of Ω and the columns of K and L are chosen in such a way that the determinants of K and L are equal to $+1$ (and not to -1). Because Ω is

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nonsingular we can apply Schur's theorem, and we find

$$\det(M) = \det(\Omega) \det \begin{vmatrix} 0 & K_0'B \\ CL_0 & D - CA^+B \end{vmatrix}, \tag{5}$$

with A^+ the Moore-Penrose inverse of A.

Second Reduction

Suppose the $r \times n$ matrix $K_0'B$ has rank s, and the $n \times r$ matrix CL_0 has rank t. In case IIa we suppose that either $t < r$ or $s < r$ or both. It follows directly from (5) that in this case $\det(M) = 0$. For case IIb we suppose that $t = s = r$.

Again case IIb has two distinct special cases. In IIb1 we have $n = t = s = r$. Thus both $K_0'B$ and CL_0 are square and nonsingular. Now

$$\det \begin{vmatrix} 0 & K_0'B \\ CL_0 & D - CA^+B \end{vmatrix} = (-1)^r \det \begin{vmatrix} K_0'B & 0 \\ D - CA^+B & CL_0 \end{vmatrix} = (-1)^r \det(K_0'B) \det(CL_0). \tag{6}$$

Thus we find in case IIb1 that

$$\det(M) = (-1)^r \det(\Omega) \det(K_0'B) \det(CL_0). \tag{7}$$

The remaining case is IIb2, in which $n > t = s = r$. We now use the two singular value decompositions

$$K_0'B = P\Psi Q_1', \tag{8}$$

$$CL_0 = R_1\Phi S'. \tag{9}$$

In (8) and (9) both P and S are square orthonormal, the matrices Q_1 and R_1 are $n \times r$ and columnwise orthonormal, we also define the $n \times (n - r)$ matrices Q_0 and R_0, which are columnwise orthonormal and satisfy $Q_0'Q_1 = R_0'R_1 = 0$. We also write T for the generalized Schur complement $D - CA^+B$. Now

$$\det \begin{vmatrix} 0 & K_0'B \\ CL_0 & T \end{vmatrix} = \det \begin{vmatrix} P' & 0 \\ 0 & R_1' \\ 0 & R_0' \end{vmatrix} \begin{vmatrix} 0 & K_0'B \\ CL_0 & T \end{vmatrix} \begin{vmatrix} S & 0 & 0 \\ 0 & Q_1 & Q_0 \end{vmatrix}$$

$$= \det \begin{vmatrix} 0 & \Psi & 0 \\ \Phi & R_1'TQ_1 & R_1'TQ_0 \\ 0 & R_0'TQ_1 & R_0'TR_0 \end{vmatrix}, \tag{10}$$

where the signs in the singular value decompositions are chosen in such a way that the determinants of P, Q, R, S are all $+1$. The $(2r) \times (2r)$ matrix in the upper left hand corner of the last expression is nonsingular, and we can apply Schur's theorem again. This gives

$$\det \begin{vmatrix} 0 & K_0'B \\ CL_0 & T \end{vmatrix} = (-1)^r \det(\Psi) \det(\Phi) \det(R_0'TR_0), \tag{11}$$

and thus, for case IIb2,

$$\det(M) = (-1)^r \det(\Omega) \det(\Psi) \det(\Phi) \det\{R_0'(D - CA^+B)Q_0\}. \tag{12}$$

A Single Formula

Provided that we use the convention that $(-1)^r = 1$ if $r = 0$, and that the determinant of a matrix of order zero is also equal to one, we can summarize all our special cases in the single formula

$$\det(M) = (-1)^r \det(K_1' A L_1) \det(K_0' B Q_1) \det(R_1' C L_0) \det\{R_0'(D - CA^+ B) Q_0\}. \tag{13}$$

A Corollary

If A is nonsingular, then M is singular if and only if $D - CA^{-1}B$ is singular. We want to make a similar statement for A singular. This answers a query of Ouellette [Note 1, p. 47]. If A is singular, then M is singular if and only if one of the matrices $K_0' BQ_1$, $R_1' CL_0$, or $R_0'(D - CA^+ B)Q_0$ is singular. This can happen if and only if case IIa or case IIb2 with $R_0'(D - CA^+ B)Q_0$ singular obtains. In case IIb1 the matrix M is nonsingular.

Applications

Van de Geer [Note 2] studies the curve defined by the algebraic equation

$$g(\mu, \lambda) = \det \begin{vmatrix} A - \mu I & C \\ C' & D - \lambda I \end{vmatrix} = 0, \tag{14}$$

with A and D real symmetric (in fact positive semidefinite). Van de Geer shows in detail that various data analytic procedures such as ridge regression estimation, oblique Procrustus rotation, principal component analysis, and various forms of canonical analysis can be interpreted in terms of the determinantal (14). Our results can be used to study the algebraic multiple valued function

$$\Lambda(\mu) = \{\lambda \in \mathbb{R} \mid g(\mu, \lambda) = 0\} \tag{15}$$

in considerable detail, especially in the points where $A - \mu I$ is singular (i.e., where μ is an eigenvalue of A). This application will be studied in more detail in another paper.

Another interesting application is to modified eigenvalue problems. We know the eigenvalues and eigenvectors of a matrix A, and we want to study the eigenvalues of $A - BB'$, either we want to compute them more efficiently using our prior knowledge of A, or we want to derive theoretical results such as perturbation bounds. The problem fits into the framework of this note because the determinantal equation

$$g(\mu) = \det(A - BB' - \mu I) = 0 \tag{16}$$

is equivalent to

$$g(\mu) = \det \begin{vmatrix} A - \mu I & B \\ B' & I \end{vmatrix} = 0. \tag{17}$$

In fact applying Schur's theorem to (17) gives

$$g(\mu) = \det(I) \det(A - \mu I - BI^{-1}B') = \det(A - BB' - \mu I). \tag{18}$$

If μ is not an eigenvalue of A then $A - \mu I$ is nonsingular and we can also apply Schur's theorem the other way around. This gives

$$g(\mu) = \det(A - \mu I) \det(I - B'(A - \mu I)^{-1}B). \tag{19}$$

If A is $m \times m$ and B is $m \times 1$, then the equation $g(\mu) = 0$ can consequently be written as

$$b'(A - \mu I)^{-1} b = 1, \tag{20}$$

where we use b instead of B, because B is now a vector. Equation (20) is familiar from the oblique Procrustus problem [cf. Ten Berge & Nevels, 1977, for a review and references], and familiar from the literature on rank-one modifications of eigenvalue problems (Golub, 1973, Bunch, Nielsen, & Sorensen, 1978]. Moreover, least squares problems with a quadratic constraint [cf. Gander, 1981, for a review and references] can also be written in this form. A recent psychometric application is Hafner [1981]. Our theory is more general, because B need not be a vector, and because $A - \mu I$ can be singular. Details of this application, however, will also be published elsewhere.

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