RATE OF CONVERGENCE
OF THE ARITHMETIC-GEOMETRIC MEAN PROCESS

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ABSTRACT. This (didactic) note gives a simple counter-example to the notion that Picard iterations converge super-linearly if and only if the sup-norm of the Jacobian at the solution is equal to zero and sub-linearly if and only if it is equal to one.

1. INTRODUCTION

Suppose S is an open subset of $\mathbb{R}^n$ and $\Gamma : S \Rightarrow S$ is a differentiable map. Assume the Picard iterations $x^{(k+1)} = \Gamma(x^{(k)})$ starting from some $x^{(0)} \in S$ converge to $x \in S$. We can derive information about the rate of convergence from the sup-norm (the eigenvalue of maximum modulus) of the derivative $\mathcal{D}\Gamma(x)$. If $\|\mathcal{D}\Gamma(x)\| = \lambda < 1$ we have linear convergence with rate λ, and if $\|\mathcal{D}\Gamma(x)\| = 0$ we have super-linear convergence [Ortega and Rheinboldt, 1970, Chapter 10]. $\|\mathcal{D}\Gamma(x)\| = 1$ often indicates sub-linear convergence. Our elementary example below, however, has $\|\mathcal{D}\Gamma(x)\| = 1$ and quadratic convergence.

2. THE ARITHMETIC-GEOMETRIC MEAN

Suppose a and b are two positive numbers. Their *arithmetic mean* is defined as $\mathbf{AM}(a,b) = \frac{1}{2}(a+b)$ and their *geometric mean* as $\mathbf{GM}(a,b) = \sqrt{ab}$.

Result 1. $\mathbf{AM}(a,b) \geq \mathbf{GM}(a,b)$ with equality if and only if $a = b$.

Proof. $0 \leq (\sqrt{a} - \sqrt{b})^2 = 2(\mathbf{AM}(a,b) - \mathbf{GM}(a,b))$. □

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From now on suppose, without loss of generality, that $a > b$. Let $a_0 = a$ and $b_0 = b$ and define the sequences

$$a_n = \mathbf{AM}(a_{n-1}, b_{n-1}), \tag{1a}$$

$$b_n = \mathbf{GM}(a_{n-1}, b_{n-1}). \tag{1b}$$

Result 2. $a_n > b_n$

Proof. From Result 1. □

Result 3. $\{a_n\}$ is a decreasing sequence, which is bounded below, and thus converges to some a_∞. $\{b_n\}$ is an increasing sequence, which is bounded above, and thus converges to some b_∞.

Proof. $a_n < \mathbf{max}(a_{n-1}, b_{n-1}) = a_{n-1}$ and $b_n > \mathbf{min}(a_{n-1}, b_{n-1}) = b_{n-1}$. Moreover $a_n > b_n > b$ and $b_n < a_n < a$. □

Result 4. $a_\infty = b_\infty$.

Proof. Take limits on both sides of (1). This gives

$$a_\infty = \mathbf{AM}(a_\infty, b_\infty),$$

$$b_\infty = \mathbf{GM}(a_\infty, b_\infty).$$

Both equations imply $a_\infty = b_\infty$. □

The common limit $a_\infty = b_\infty$ is called the *arithmetic-geometric mean* of a and b, written as $\mathbf{AGM}(a, b)$. The arithmetic-geometric mean was studied by Legendre and Gauss, and it has fascinating applications in many areas of mathematics and numerical analysis. There are excellent reviews of these applications in Carlson [1971], Cox [1984], and Almqvist and Berndt [1988].

Result 5. b $< \mathbf{GM}(a, b) < \mathbf{AGM}(a, b) < \mathbf{AM}(a, b) < a$

Proof. $\mathbf{AGM}(a, b) = a_\infty < a_1 = \mathbf{AM}(a, b) < a_0 = a$ and $\mathbf{AGM}(a, b) = b_\infty > b_1 = \mathbf{GM}(a, b) > b_0 = b$. □

For another proof of the convergence to a common limit we define the sequence $\delta_n = a_n - b_n$. It should be noted that δ_n is a reasonable way to measure distance to the solution, since

$$|a_n - \mathbf{AGM}(a,b)| + |b_n - \mathbf{AGM}(a,b)| = \\ a_n - \mathbf{AGM}(a,b) + \mathbf{AGM}(a,b) - b_n = \delta_n.$$

Result 6. $\{\delta_n\}$ is a decreasing sequence bounded below by zero, and thus converges to some $\delta_\infty \geq 0$.

Proof. Since $a_n < a_{n-1}$ and $b_n > b_{n-1}$ we have $\delta_n = a_n - b_n < a_{n-1} - b_{n-1} = \delta_{n-1}$. Moreover $\delta_n > 0$ for all n. □

Result 7. $\delta_\infty = 0$

Proof.

$$\text{(2a)} \qquad \delta_n = \mathbf{AM}(a_{n-1}, b_{n-1}) - \mathbf{GM}(a_{n-1}, b_{n-1}) = \frac{1}{2}(\sqrt{a_{n-1}} - \sqrt{b_{n-1}})^2,$$

$$\text{(2b)} \quad \delta_{n-1} = a_{n-1} - b_{n-1} = (\sqrt{a_{n-1}} - \sqrt{b_{n-1}})(\sqrt{a_{n-1}} + \sqrt{b_{n-1}}),$$

and thus $\delta_n < \frac{1}{2}\delta_{n-1}$ It follows that $0 < \delta_n < (\frac{1}{2})^n \delta_0$ and thus $\lim_{n\to\infty} \delta_n = 0$. □

The proof shows that convergence of $\{\delta_n\}$ is faster than that of a geometric sequence with radius $\frac{1}{2}$. But we can be more precise.

Result 8. Convergence of the sequence $\{\delta_n\}$ to zero is superlinear, i.e.

$$\lim_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = 0.$$

Proof. From Equations (2)

$$\frac{\delta_n}{\delta_{n-1}} = \frac{1}{2} \frac{\sqrt{a_{n-1}} - \sqrt{b_{n-1}}}{\sqrt{a_{n-1}} + \sqrt{b_{n-1}}} \to 0.$$

□

In fact, we can be even more precise.

Result 9. Convergence of the sequence $\{\delta_n\}$ to zero is quadratic.

$$\lim_{n\to\infty} \frac{\delta_n}{\delta_{n-1}^2} = \frac{1}{8} \frac{1}{\mathbf{AGM}(a,b)}.$$

Proof. From Equations (2)

$$\frac{\delta_n}{\delta_{n-1}^2}=\frac{1}{2}\frac{1}{(\sqrt{a_{n-1}}+\sqrt{b_{n-1}})^2}\rightarrow\frac{1}{8}\frac{1}{\mathbf{AGM}(a,b)}.$$

□

In a sense, the sequences $\{a_n\}$ and $\{b_n\}$ converge equally fast.

Result 10. $(a_n-a_{n-1})\sim-(b_n-b_{n-1})$, i.e.

$$\lim_{n\rightarrow\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=-1.$$

Proof.

$$a_n-a_{n-1}=-\frac{1}{2}(\sqrt{a_{n-1}}-\sqrt{b_{n-1}})(\sqrt{a_{n-1}}+\sqrt{b_{n-1}}),$$
$$b_n-b_{n-1}=\sqrt{b_{n-1}}(\sqrt{a_{n-1}}-\sqrt{b_{n-1}}).$$

and thus

$$\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=-\frac{1}{2}\frac{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}{\sqrt{b_{n-1}}}\rightarrow-1.$$

□

3. Counterexample

Equation (1) defines a mapping $\Gamma:\mathbb{R}^2\Rightarrow\mathbb{R}^2$. The derivative of this mapping is

$$\mathcal{D}\Gamma(a,b)=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}\sqrt{\frac{b}{a}}&\frac{1}{2}\sqrt{\frac{a}{b}}\end{bmatrix},$$

and thus

$$\mathcal{D}\Gamma(\mathbf{AGM}(a,b),\mathbf{AGM}(a,b))=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix},$$

which has eigenvalues one and zero.

The fact that $\|\mathcal{D}\Gamma(\mathbf{AGM}(a,b),\mathbf{AGM}(a,b))\|_\infty=1$ seems to suggest sublinear convergence, while in fact we know convergence is quadratic. If y_n is the two-element vector with elements a_n-a_{n-1} and b_n-b_{n-1}, normalized to length one, then Result 10 shows that y_n converges to a vector with elements -1 and $+1$. This eigenvector corresponds with the smallest eigenvalue of the Jacobian at the solution, and that smallest eigenvalue is equal to zero.

References

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Appendix A. Code

```
agm<-function(a,b,eps=1e-8,itmax=1000,verbose=TRUE)
{
xold<-max(a,b); yold<-min(a,b); dold<-xold-yold; itel<-1
repeat {
    xnew<-(xold+yold)/2; ynew<-sqrt(xold*yold)
    dnew<-xnew-ynew; rat1<-dnew/dold; rat2<-dnew/(dold^2)
    if (verbose) cat(
        "Iteration: ",formatC(itel,width=3, format="d"),
        "old: ",formatC(c(xold,yold,dold),digits=8,
                 width=12,format="f"),
        "old: ",formatC(c(xnew,ynew,dnew),digits=8,
                 width=12,format="f"),
        "rat: ",formatC(c(rat1,rat2),digits=8,
                 width=12,format="f"),
        "\n")
    if ((dnew < eps) || (itel == itmax))
        return(c(xnew,ynew))
    xold<-xnew; yold<-ynew; dold<-dnew; itel<-itel+1
    }
}
```

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