THE MULTIWAY PACKAGE

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ABSTRACT. This note documents R code for the CANDECOMP and TUCKER generalizations of the Singular Value Decomposition to multiway arrays. Alternating Least Squares algorithms are used to generate a convergent sequence of low-rank approximations.

1. CANDECOMP

```
1  candecomp<-function(a,x,ortho=rep(FALSE,length(x)),
2       itmax=1000,eps=1e-6,verbose=FALSE)
```

The two leading arguments of the function `candecomp()` are an $n_1 \times n_2 \times \cdots \times n_m$ array `a` and a list of m matrices `x`, where `x[[i]]` has $n_i \times p$ elements. The function minimizes the least squares loss function over the `x[[i]]`, using the original values are starting values.

$$\sigma(x) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m}(a_{i_1 i_2 \cdots i_m} - \sum_{s=1}^{p} x^1_{i_1 s} x^2_{i_2 s}.\cdots x^m_{i_m s})^2.$$

There is an additional parameter `ortho`, a logical vector of length m. If `ortho[i]` is `TRUE` then `x[[i]]` is required to be columnwise orthonormal. The additional arguments `itmax`, `eps` and `verbose` set the iteration parameters.

2. TUCKER

```
1  tucker<-function(a,x,ident=rep(FALSE,length(x)),
2       itmax=1000,eps=1e-6,verbose=FALSE)
```

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The leading arguments `a` and `x` are the same as before, except that now the `x[[i]]` are $n_i \times p_i$ columnwise orthonormal matrices. The loss function is

$$\sigma(x,b) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_m=1}^{n_m}(a_{i_1i_2\cdots i_m} - \sum_{s_1=1}^{p_1}\sum_{s_2=1}^{p_2}\cdots\sum_{s_m=1}^{p_m} b_{s_1s_2\cdots s_m}x^1_{i_1s_1}x^2_{i_2s_2}\cdots x^m_{i_ms_m})^2.$$

The $p_1 \times p_2 \times \cdots \times p_m$ array `b` is called the *core array*. There is an additional parameter `ident`, a logical vector of length m. If `ident[i]` is `TRUE` then matrix `x[[i]]` is restricted to be the $n_i \times n_i$ identity matrix.

Note that we can also write

$$\sigma(x,b) = \|a - (X_1 \otimes X_2 \otimes \cdots \otimes X_m)b\|^2,$$

where a and b are the $vec(\bullet)$ of the arrays, $\otimes$ is the Kronecker product, and $\|\bullet\|$ is the Frobenius norm.

3. UTILITIES

The package includes functions that compute generalized Hadamard products, generalized outer products and generalized Kronecker products on lists of matrices. A Procrustus approximation routine is also included.

4. THREE-WAY

In the important three-way case `tucker()` with `ident[3]=TRUE` fits the approximation $A_k = XB_kY'$, while `candecomp()` fits the same approximation $A_k = XB_kY'$, with the B_k restricted to be diagonal.

5. CODE

```
#
#   multiway package
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#
# version 0.1.0, 2008-03-27,            first release
# version 0.2.0, 2008-03-28,            added tucker and constraints
#

candecomp<-function(a,x,ortho=rep(FALSE,length(x)),itmax=1000,eps=1e-6,verbose=
    FALSE) {
        ndam<-dim(a)
        nard<-length(ndam)
        ndim<-ncol(x[[1]])
        for (k in 1:nard) if (ortho[k]) x[[k]]<-procrustus(x[[k]])
        c<-lapply(x,crossprod)
        oloss<-candeValue(a,x)$loss
        itel<-1
        repeat {
                for (k in 1:nard) {
                        cc<-arrHadamard(c[-k])
                        for (p in 1:ndim) {
                                y<-lapply(x[-k],function(z) z[,p])
                                b<-arrOuter(y)
                                x[[k]][,p]<-apply(a,k,function(z) sum(z*b))
                                }
                        if (ortho[k]) x[[k]]<-procrustus(x[[k]])
                                else x[[k]]<-t(solve(cc,t(x[[k]])))
                        c[[k]]<-crossprod(x[[k]])
                        }
                        cval<-candeValue(a,x)
                        nloss<-cval$loss; ahat<-cval$ahat
                if (verbose)
                        cat("Iteration: ",formatC(itel,digits=3,width=3),
                                "Old Loss: ",formatC(oloss,digits=6,width=10,
                                    format="f"),
                                "New Loss: ",formatC(nloss,digits=6,width=10,
                                    format="f"),
                                "\n")
                if ((itel == itmax) | ((oloss - nloss) < eps)) break()
                itel<-itel+1; oloss<-nloss
                }
        return(list(x=x,ahat=ahat,loss=nloss))
        }

candeValue<-function(a,x) {
        ndim<-ncol(x[[1]])
        ahat<-array(0,dim(a))
        for (p in 1:ndim) {
                y<-sapply(x,function(z) z[,p])
                b<-arrOuter(y)
```

```
                ahat<-ahat+b
                }
        loss<-sum((a-ahat)^2)
        return(list(ahat=ahat,loss=loss))
        }

tucker<-function(a,x,ident=rep(FALSE,length(x)),itmax=1000,eps=1e-6,verbose=FALSE)
      {
        ndam<-dim(a)
        for (k in 1:nard) if (ident[k]) x[[k]]<-diag(ndam[k])
        ndbm<-sapply(x,function(z) ncol(z))
        nard<-length(ndam)
        rard<-rev(1:nard)
        x<-lapply(x,procrustus)
        xx<-arrKronecker(x)
        aa<-as.vector(aperm(a,rard))
        bb<-colSums(aa*xx)
        b<-aperm(array(bb,rev(ndbm)),rard)
        ahat<-aperm(array(drop(xx%*%bb),rev(ndam)),rard)
        oloss<-sum((a-ahat)^2)
        itel<-1
        repeat {
                for (k in 1:nard) {
                        if (!ident[k]) {
                                z<-crossprod(flatten(a,k),arrKronecker(x[-k]))
                                     %*%flatten(b,k)
                                x[[k]]<-procrustus(z)
                                }
                        }
                xx<-arrKronecker(x)
                aa<-as.vector(aperm(a,rard))
                bb<-colSums(aa*xx)
                b<-aperm(array(bb,rev(ndbm)),rard)
                ahat<-aperm(array(drop(xx%*%bb),rev(ndam)),rev(1:nard))
                nloss<-sum((a-ahat)^2)
                if (verbose)
                        cat("Iteration: ",formatC(itel,digits=3,width=3),
                                "Old Loss: ",formatC(oloss,digits=6,width=10,
                                     format="f"),
                                "New Loss: ",formatC(nloss,digits=6,width=10,
                                     format="f"),
                                "\n")
                if ((itel == itmax) | ((oloss - nloss) < eps)) break()
                itel<-itel+1; oloss<-nloss
                }
        return(list(x=x,b=b,ahat=ahat,loss=nloss))
        }

arrHadamard<-function(c,fun=function(x,y) x*y) {
        nmat<-length(c)
        if (nmat == 0) stop("empty argument in arrHadamard")
        res<-c[[1]]
        if (nmat == 1) return(res)
```

```
114         for (i in 2:nmat) res<-fun(res,c[[i]])
115         return(res)
116         }
117
118 arrOuter<-function(x,fun="*") {
119         nmat<-length(x)
120         if (nmat == 0) stop("empty argument in arrOuter")
121         res<-x[[1]]
122         if (length(x) == 1) return(res)
123         for (i in 2:nmat) res<-outer(res,x[[i]],fun)
124         return(res)
125         }
126
127 arrKronecker<-function(x,fun="*") {
128         nmat<-length(x)
129         if (nmat == 0) stop("empty argument in arrKronecker")
130         res<-x[[1]]
131         if (length(x) == 1) return(res)
132         for (i in 2:nmat) res<-kronecker(res,x[[i]],fun)
133         return(res)
134         }
135
136 flatten<-function(a,k) {
137         nard<-rev(1:(length(dim(a))-1))
138         apply(a,k,function(z) as.vector(aperm(z,nard)))
139 }
140
141 procrustus<-function(x) {
142         res<-svd(x)
143         return(tcrossprod(res$u,res$v))
144 }
```

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