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THE EFFECT OF A CUT-OFF STRATEGY ON THE ALPHA-PRIME MEASURE

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In probabilistic concept learning experiments (de Klerk & Oppe 1966) stimuli are constructed in such a way that their projections on the loglikelihood axis X are distributed as follows (de Leeuw 1968)

$$f(x) = \frac{P}{\sqrt{2\pi\alpha}} \exp\left\{-\frac{1}{2}\left(\frac{x+\frac{1}{2}\alpha}{\sqrt{\alpha}}\right)^2\right\} + \frac{(1-P)}{\sqrt{2\pi\alpha}} \exp\left\{-\frac{1}{2}\left(\frac{x-\frac{1}{2}\alpha}{\sqrt{\alpha}}\right)^2\right\}$$

Suppose that the subject classifies the stimuli according to the following rule: a stimulus is a positive instance of the concept if $x > a$, otherwise it is a negative instance of the concept. This strategy generates two distributions on the X-axis

$$g_1(x) = \begin{cases} K_1 f(x) & (x < a) \\ 0 & (x > a) \end{cases}$$

$$g_2(x) = \begin{cases} K_2 f(x) & (x > a) \\ 0 & (x < a) \end{cases}$$

Now define

$$\alpha' = \frac{\eta_2 - \eta_1}{\sqrt{\frac{1}{2}(\gamma_1^2 + \gamma_2^2)}}$$

with

$$\eta_i = \int_{-\infty}^{+\infty} x g_i(x)dx$$

and

$$\gamma_i^2 = \int_{-\infty}^{+\infty} (x-\eta_i)^2 g_i(x)dx$$

The parameter α' is a function of the a priori probability P, the Mahanalobis distance α and the place of the cut-off value a on the continuum.

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Some definitions

$$k = (2\pi)^{-\frac{1}{2}}$$

$$z_1 = \frac{a + \frac{1}{2}\alpha}{\sqrt{\alpha}}$$

$$z_2 = \frac{a - \frac{1}{2}\alpha}{\sqrt{\alpha}}$$

$$y_1 = k \exp(-\tfrac{1}{2}z_1^2)$$

$$y_2 = k \exp(-\tfrac{1}{2}z_2^2)$$

$$p_1 = k \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)\, dx$$

$$p_2 = k \int_{-\infty}^{z_2} \exp(-\tfrac{1}{2}x^2)\, dx$$

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$$K_1 = \frac{1}{\int_{-\infty}^{a} f(x)dx} = \frac{1}{Pp_1 + (1-P)p_2}$$

Because

$$\int_{-\infty}^{a} f(x)dx = kP \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)dx + k(1-P) \int_{-\infty}^{z_2} \exp(-\tfrac{1}{2}x^2)dx =$$

$$= Pp_1 + (1-P)p_2$$

Of course

$$\int_{a}^{+\infty} f(x)dx = 1 - \int_{-\infty}^{a} f(x)dx = 1-Pp_1 + (1-P)p_2 = P(1-p_1) +$$

$$+ (1-P)(1-p_2)$$

which implies

$$K_2 = \frac{1}{P(1-p_1) + (1-P)(1-p_2)}$$

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$$\eta_1 = \int_{-\infty}^{a} x g_1(x)dx = K_1 \int_{-\infty}^{a} x f(x) dx =$$

$$= K_1 P \int_{-\infty}^{a} x f_1(x)dx + K_1(1-P) \int_{-\infty}^{a} x f_2(x)dx =$$

$$= K_1 P T_1 + K_1(1-P)T_2$$

Where

$$T_1 = k \int_{-\infty}^{z_1} (x\sqrt{\alpha} - \tfrac{1}{2}\alpha)\exp(-\tfrac{1}{2}x^2)dx =$$

$$= k\sqrt{\alpha} \int_{-\infty}^{z_1} x\exp(-\tfrac{1}{2}x^2)dx - \tfrac{1}{2}k\alpha \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)dx =$$

$$= -y_1\sqrt{\alpha} - \tfrac{1}{2}\alpha\, p_1$$

$$T_2 = k \int_{-\infty}^{z_2} (x\sqrt{\alpha} + \tfrac{1}{2}\alpha)\exp(-\tfrac{1}{2}x^2)dx =$$

$$= -y_2\sqrt{\alpha} + \tfrac{1}{2}\alpha\, p_2$$

In the same way

$$\eta_2 = K_2 P T_3 + K_2(1-P)T_4$$

where

$$T_3 = y_1\sqrt{\alpha} - \tfrac{1}{2}\alpha(1-p_1) = y_1\sqrt{\alpha} - \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha\, p_1$$

$$T_4 = y_2\sqrt{\alpha} + \tfrac{1}{2}\alpha(1-p_2) = y_2\sqrt{\alpha} + \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha\, p_2 .$$

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$$\gamma_1^2 + \eta_1^2 = \int_{-\infty}^{a} x^2 g_1(x)dx =$$

$$= K_1 P \int_{-\infty}^{a} x^2 f_1(x)dx + K_1(1-P) \int_{-\infty}^{a} x^2 f_2(x)dx =$$

$$= K_1 P T_5 + K_1(1-P)T_6$$

where

$$T_5 = k \int_{-\infty}^{z_1} (x\sqrt{\alpha} - \tfrac{1}{2}\alpha)^2 \exp(-\tfrac{1}{2}x^2)dx =$$

$$= k\,\alpha \int_{-\infty}^{z_1} x^2 \exp(-\tfrac{1}{2}x^2)dx + \tfrac{1}{4}k\,\alpha^2 \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)dx +$$

$$= -k\alpha\sqrt{\alpha} \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)dx$$

Now

$$k \int_{-\infty}^{z_1} x^2 \exp(-\tfrac{1}{2}x^2)dx = -z_1 y_1 + k \int_{-\infty}^{z_1} \exp(-\tfrac{1}{2}x^2)dx = -z_1 y_1 + p_1.$$

Thus

$$T_5 = -z_1 y_1 \alpha + p_1 \alpha + \tfrac{1}{4}\alpha^2 p_1 + y_1 \alpha \sqrt{\alpha}$$

$$T_6 = -z_2 y_2 \alpha + p_2 \alpha + \tfrac{1}{4}\alpha^2 p_2 - y_2 \alpha \sqrt{\alpha}$$

In the same way

$$\gamma_2^2 + \eta_2^2 = K_2 P T_7 + K_2 (1-P) T_8$$

where

$$T_7 = \alpha + \alpha\, z_1 y_1 - \alpha\, p_1 - y_1 \alpha \sqrt{\alpha} + \tfrac{1}{4}\alpha^2 - \tfrac{1}{4}\alpha^2 p_1$$

$$T_8 = \alpha + \alpha\, z_2 y_2 - \alpha\, p_2 + y_2 \alpha \sqrt{\alpha} + \tfrac{1}{4}\alpha^2 - \tfrac{1}{4}\alpha^2 p_2$$

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In the special case that $a = 0$ and $P=\frac{1}{2}$ we have the following simplifications:

$$z_1 = -z_2 =_{\text{def}} z$$

$$p_1 = 1-p_2 =_{\text{def}} p$$

$$K_1 = K_2 = 2$$

$$y_1 = y_2 =_{\text{def}} y$$

$$\eta_1 = -\eta_2 = T_1 + T_2 = -2y\sqrt{\alpha} - \alpha p + \tfrac{1}{2}\alpha$$

$$\gamma_1^2 + \eta_1^2 = \gamma_2^2 + \eta_2^2 = T_7 + T_8 = \tfrac{1}{4}\alpha^2 + \alpha$$

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Take the limit case $\sqrt{\alpha} \to \infty$, then $y_1, y_2 \to 0$; $p_1 \to 1$; $p_2 \to 0$; $K_1 \to P^{-1}$; $K_2 \to (1-P)^{-1}$; $T_1 \to -\frac{1}{2}\alpha$; $T_4 \to +\frac{1}{2}\alpha$; $T_2, T_3, T_6, T_7 \to 0$; $T_5, T_8 \to \frac{1}{4}\alpha^2 + \alpha$.

In other words $\alpha' \to \sqrt{\alpha}$.

$\lim_{d' \to 0} \alpha' = \sqrt{\frac{8}{\pi - 2}} \sim 2.6472$

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In table I and figure I values of α' for various values of d' are given, in the special case that a = 0. Obviously the conclusion from these results is that for small values of d' (<1) the α'-measure does not even give ordinal information about the distance of the populations. For intermediate values of d' ($1 < d' < 5$) the α'-scale is a strictly monotone transformation of the d'-scale, and for $d' > 5$ the two measures are numerically identical (that is to say: within the usual limits of precision, see section 7).

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In table II the α'-values are given for some values of a, in the special case that d' = 2. The conclusion seems to be, that the location of the cut-off value (i.e. the response bias) has but little effect on α', even in the most perverse cases (a = -4). Results for the more plausible case that $a = -\frac{1}{4}d'$ are given in table III for various values of d'.

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In the case of a cut-off strategy the confusion matrix for the two concepts and the two responses becomes

		Y	N
concept	A	$1-p_2$	p_2
	B	$1-p_1$	p_1

(Note that the a priori probabilities of the concepts are irrelevant for this confusion matrix). The two parameters of Luce's choice model for forced-choice detection experiments (Luce 1959, 1963) are given by

$$\eta = \left[\frac{p(N/A)p(Y/B)}{p(Y/A)p(N/B)}\right]^{\frac{1}{2}} = \left[\frac{p_2(1-p_1)}{p_1(1-p_2)}\right]^{\frac{1}{2}}$$

$$b = \left[\frac{p(N/A)p(N/B)}{p(Y/A)p(Y/B)}\right]^{\frac{1}{2}} = \left[\frac{p_1 p_2}{(1-p_1)(1-p_2)}\right]^{\frac{1}{2}}$$

As is pointed out by Luce the negative logarithm of η can be compared with the d'-parameter of signal detection theory. The Tanner & Swets β-parameter is defined as

$$\beta = \frac{f_1(a)}{f_2(a)} = \exp\left\{-\tfrac{1}{2}\left(\frac{a+\frac{1}{2}\alpha}{\sqrt{\alpha}}\right)^2+\tfrac{1}{2}\left(\frac{a-\frac{1}{2}\alpha}{\sqrt{\alpha}}\right)^2\right\} = \exp(-a)$$

or

$$-\ln\beta = a$$

This makes it reasonable to compare the logarithm of b with a. Both η and b are measured on ratio scales, which means that the unit of measurement is arbitrary. We have taken all logarithms to the base 10, and we have scaled $-\log\eta$ and log b afterwards by minimizing some least squares criterion.

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In the special case that a=0 we have the following simplifications

$$\eta = \left[\frac{p_2^2}{p_1^2}\right]^{\frac{1}{2}} = \frac{p_2}{p_1} = \frac{1-p}{p}$$

$$b = \left[\frac{p_1(1-p_1)}{p_1(1-p_1)}\right]^{\frac{1}{2}} = 1$$

which means that log b = 0 = $-\ln\beta$ = a (as it should be).

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In table IV and figure II values of $-A\log\eta$ for various values of d' are given, in the special case that a = 0. If we compare these results with those in table I and figure I, it is clear that η does a better job than α', especially in the lower regions of the scale. In table V and figures IIIa and IIIb, the same cases are treated as in table II. These figures suggest that η is more sensitive to extreme response bias, but in the more moderate cases treated in table III this effect is neglectable (table VI, figures IVa and IVb).

TABLE I: CASE a=0 AND P=$\frac{1}{2}$

d'	η_2	γ	α'
0.1	0.0799	0.0600	2.6600
0.2	0.1604	0.1212	2.6469
0.3	0.2422	0.1825	2.6542
0.4	0.3254	0.2460	2.6455
0.5	0.4114	0.3105	2.6499
0.6	0.5001	0.3772	2.6516
0.7	0.5923	0.4463	2.6543
0.8	0.6888	0.5176	2.6615
0.9	0.7895	0.5922	2.6663
1.0	0.8958	0.6690	2.6780
1.5	1.5186	1.0998	2.7616
2.0	2.3332	1.5988	2.9187
2.5	3.3780	2.1459	3.1483
3.0	4.6758	2.7179	3.4407
5.0	12.5200	4.9497	5.0589
10.0	50.0000	10.0000	10.0000

TABLE II: CASE d'=2 AND P=$\frac{1}{2}$

a	η_1	η_2	γ_1^2	γ_2^2	$\bar{\gamma}$ 1)	α'
-4.0000	-5.0475	0.4389	0.7827	6.2198	1.8712	2.9320
-3.0000	-4.2701	0.7974	1.0657	5.2545	1.7777	2.8506
-2.0000	-3.5582	1.2594	1.4434	4.2540	1.6878	2.8544
-1.0000	-2.9178	1.7818	1.9304	3.3320	1.6221	2.8972
0.0000	-2.3332	2.3332	2.5562	2.5562	1.5988	2.9187

TABLE III: CASE a=$-\frac{1}{4}$d' AND P=$\frac{1}{2}$

d'	η_1	η_2	γ_1^2	γ_2^2	$\bar{\gamma}$ 1)	α'
0.5	-0.4517	0.3726	0.0930	0.0985	0.3095	2.6633
1.0	-1.0589	0.7439	0.3890	0.5139	0.6719	2.6831
1.5	-1.8747	1.1895	0.9845	1.4790	1.1099	2.7608
2.0	-2.9178	1.7818	1.9305	3.3320	1.6221	2.8972
3.0	-5.6311	3.6430	5.0302	11.1356	2.8430	3.2621
10.0	-50.1740	49.5560	95.6711	131.0593	10.6474	9.3666

1) $\bar{\gamma} = \sqrt{\frac{1}{2}(\gamma_1^2 + \gamma_2^2)}$

TABLE IV: CASE a=0

d'	P	η	$-A_1 \log \eta$
0.1	0.5199	0.9234	0.0847
0.2	0.5398	0.8525	0.1697
0.3	0.5596	0.7870	0.2547
0.4	0.5793	0.7262	0.3402
0.5	0.5987	0.6703	0.4254
0.6	0.6179	0.6184	0.5111
0.7	0.6368	0.5704	0.5971
0.8	0.6554	0.5258	0.6838
0.9	0.6736	0.4846	0.7683
1.0	0.6915	0.4461	0.8587
1.5	0.7734	0.2930	1.3056
2.0	0.8413	0.1886	1.7744
2.5	0.8944	0.1181	2.2723
3.0	0.9332	0.0716	2.8045
5.0	0.9938	0.0062	5.4066
10.0	1.0000	0.0000	∞

TABLE V: CASE d'=2

a	P_1	P_2	η	b	$-A_2 \log \eta$	$A_3 \log b$
-4.0000	0.1587	0.0013	0.0831	0.0141	2.4577	-4.0843
-3.0000	0.3085	0.0062	0.1183	0.0529	2.1087	-3.0377
-2.0000	0.5000	0.0228	0.1526	0.1526	1.8571	-1.8016
-1.0000	0.6915	0.0668	0.1786	0.3942	1.7018	-0.8922
0.0000	0.8413	0.1587	0.1887	1.0000	1.6474	0.0000

TABLE VI: CASE a = $-\frac{1}{4}d'$

d'	P_1	P_2	η	b	$-A_4 \log \eta$	$A_5 \log b$
0.5	0.5500	0.3540	0.6696	0.8184	0.4474	-0.2498
1.0	0.5987	0.2266	0.4432	0.6611	0.9076	-0.5100
1.5	0.6460	0.1304	0.2867	0.5231	1.3935	-0.8080
2.0	0.6915	0.0668	0.1787	0.4005	1.9078	-1.1411
3.0	0.7734	0.0122	0.0600	0.2052	3.1377	-1.9750
10.0	0.9938	0.0000	0.0000	0.0000	∞	$-\infty$

Note:

$A_1 = 2.4491$; $A_2 = 2.2068$; $A_3 = 2.2748$; $A_4 = 2.8715$; $A_5 = 2.5681$

cf: $1/\log e = 2.3003$.

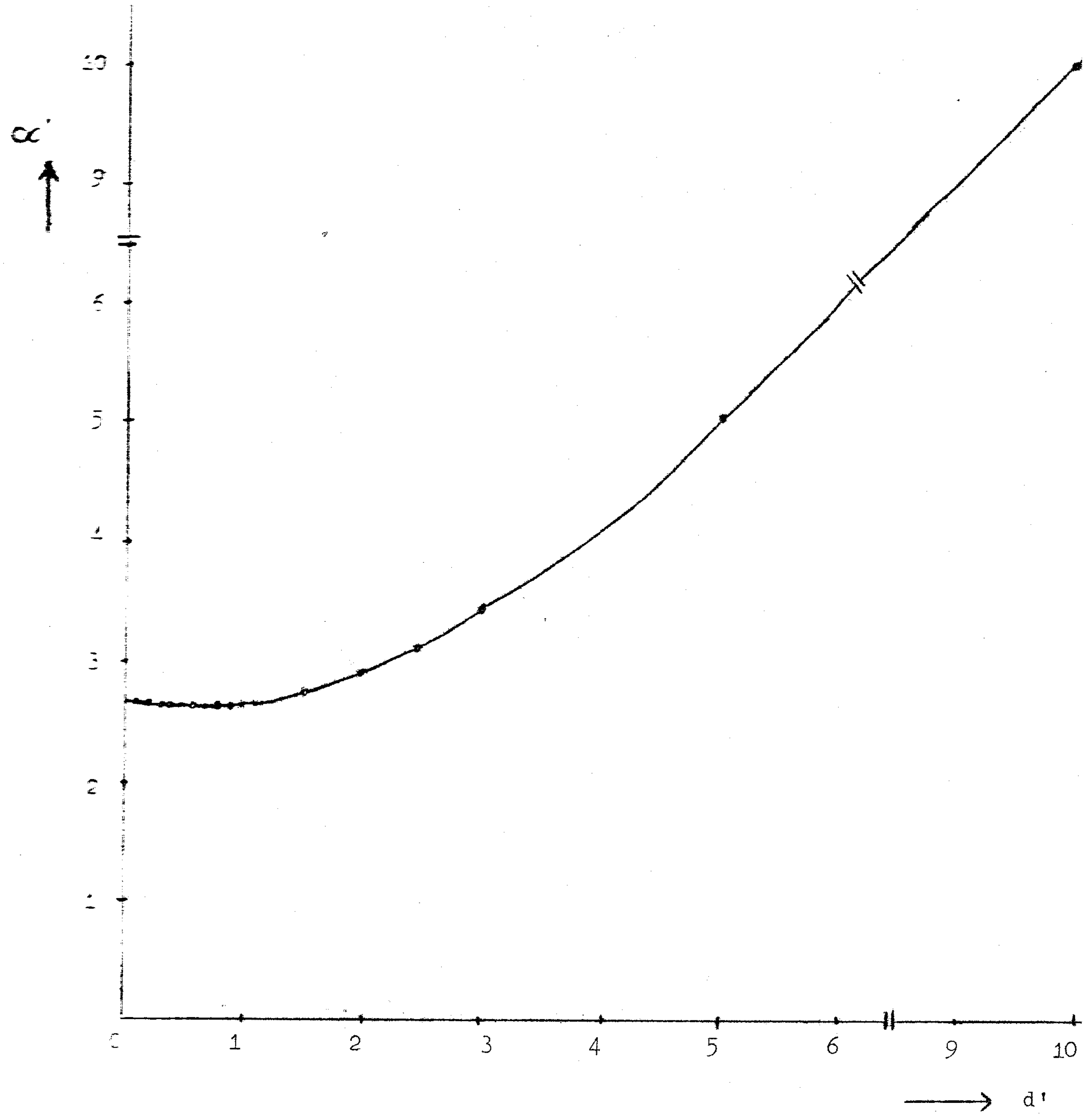

Figure I.

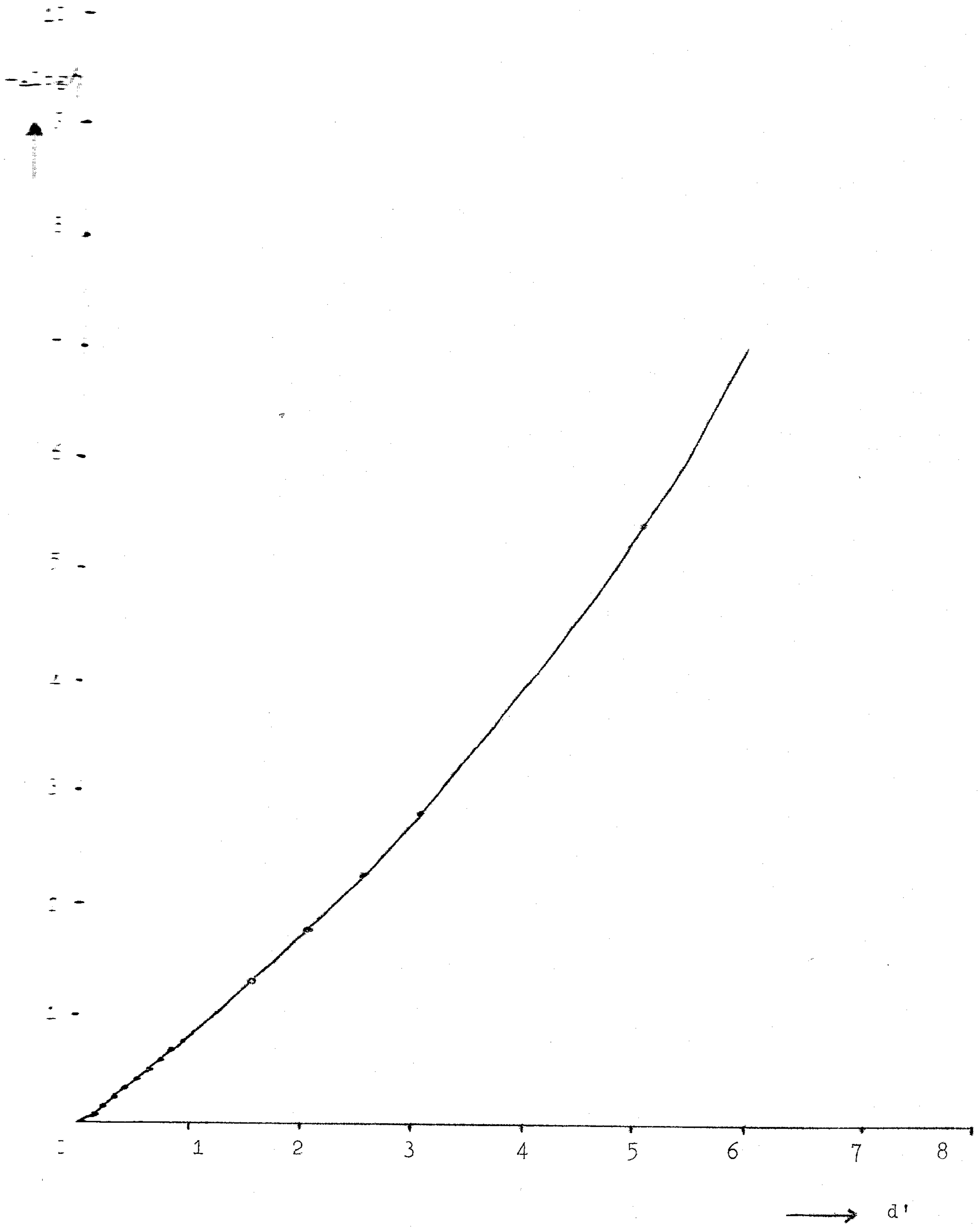

Figure II.

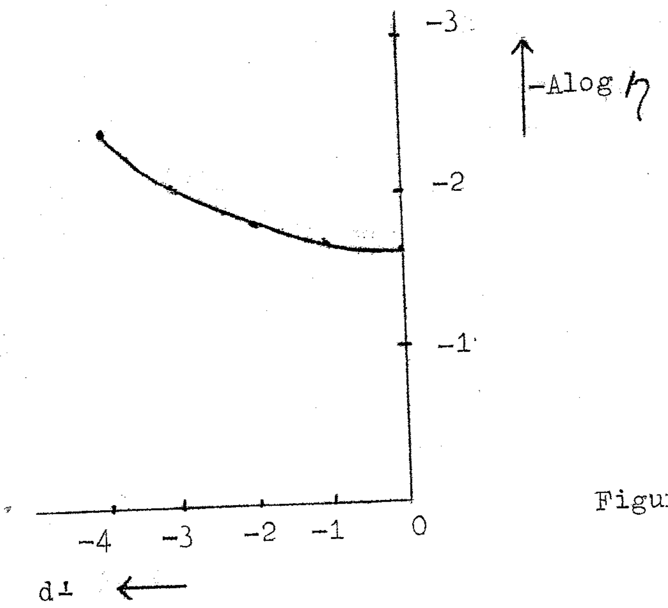

Figure IIIa.

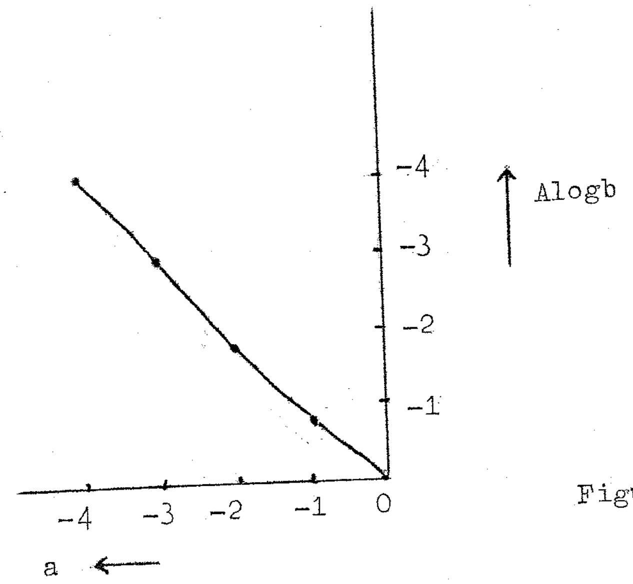

Figure IIIb.

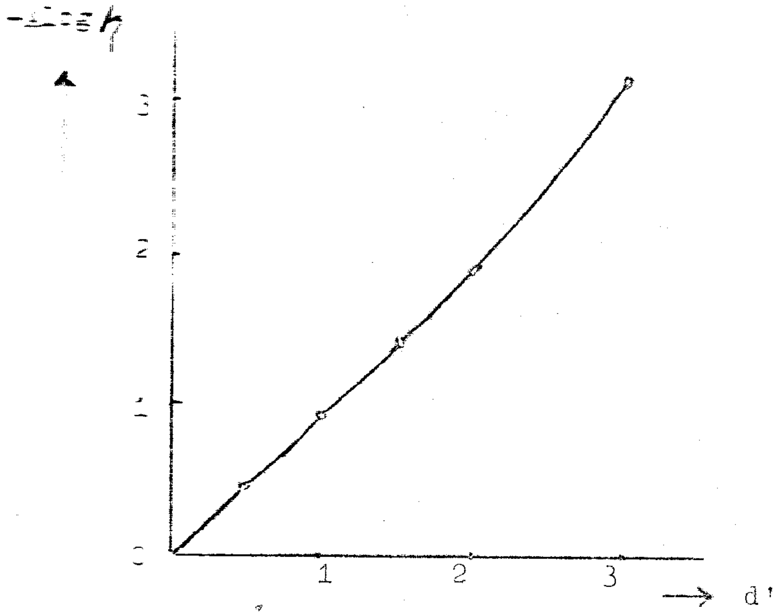

Figure IVa.

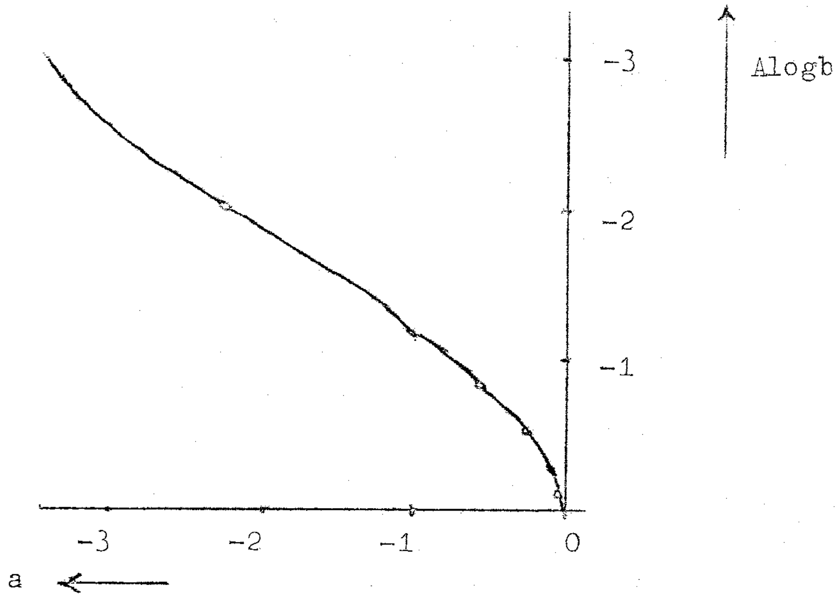

Figure IVb.

LITERATURE

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