THE SPEARMAN MODEL

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1. Spearman Structures

Suppose $\mathcal{Y}$ is a linear space with inner product $< \bullet, \bullet >$ and with unit sphere $\mathcal{S} \stackrel{\Delta}{=} \{y \in \mathcal{Y} \mid < y, y >= 1\}$.

Definition 1.1. Sequence $\{y_j \in \mathcal{Y}\}_{1 \leq j \leq m}$ is a *Spearman sequence* if there exist

- $u \in \mathcal{S}$,
- $\{e_j \in \mathcal{S}\}_{1 \leq j \leq m}$,
- $\{\alpha_j \in \mathbb{R}\}_{1 \leq j \leq m}$,
- $\{\delta_j \in \mathbb{R}\}_{1 \leq j \leq m}$,

such that

- $y_j = \alpha_j u + \delta_j e_j$ for all $1 \leq j \leq m$,
- $< u, e_j >= 0$ for all $1 \leq j \leq m$,
- $< e_j, e_\ell >= 0$ for all $1 \leq j < \ell \leq m$.

Definition 1.2. A real symmetric matrix C of order m is a *Spearman matrix* if there exist

- $\{\alpha_j \in \mathbb{R}\}_{1 \leq j \leq m}$,
- $\{\delta_j \in \mathbb{R}\}_{1 \leq j \leq m}$,

such that

$$c_{j\ell} = \begin{cases} \alpha_j^2 + \delta_j^2 & \text{for all } 1 \leq j \leq m, \\ \alpha_j \alpha_\ell & \text{for all } 1 \leq j \neq \ell \leq m. \end{cases}$$

2. Fundamental Theorem of Factor Analysis

2.1. Existence.

Theorem 2.1. *If $\{y_j \in \mathcal{Y}\}_{1 \leq j \leq m}$ is a Spearman sequence, then C with elements $c_{j\ell} =< y_j, y_\ell >$ is a Spearman matrix. Conversely, if C is a Spearman matrix, then there exists a Spearman sequence $\{y_j \in \mathcal{Y}\}_{1 \leq j \leq m}$ with $c_{j\ell} =< y_j, y_\ell >$.*

Proof. The first part is a simple calculation. □

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2.2. **Indeterminacy.**

3. PROPERTIES OF SPEARMAN MATRICES

3.1. **Canonical Form.**

Definition 3.1. A Spearman matrix is *regular* if

- $\alpha_j > 0$ for all $1 \leq j \leq m$,
- $\delta_1^2 > \delta_2^2 > \cdots > \delta_m^2$.

A regular Spearman matrix is *complete* if $\delta_m^2 > 0$, otherwise it is *incomplete*.

Theorem 3.1. *Each Spearman matrix is (orthogonally) similar to the direct sum of a regular Spearman matrix and a diagonal matrix.*

Proof. We first permute rows and columns of the Spearman matrix C such that those with both $\alpha_j = 0$ and $\delta_j^2 = 0$ come last. Suppose there are m_{00} of these.

Then permute again so that the, say, m_{01} rows and columns with $\alpha_j = 0$ and $\delta_j^2 > 0$ come before these. We then have

$$C \sim \begin{bmatrix} \tilde{C} & 0 & 0 \\ 0 & \tilde{\Delta}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now look at the submatrix $\tilde{C}$, for which all $\alpha_j \neq 0$. Permute again to make the δ_j^2 non-increasing along the diagonal. Suppose the δ_j^2 have r different values, and that value δ_s^2 has multiplicity k_s. Write α_s for the subvector of α corresponding to δ_s^2.

Construct the r orthonormal matrices L_s, of order k_s, whose first columns are equal to $\alpha_s/\|\alpha_s\|$, and whose remaining columns are orthogonal to α_s, and to each other. Premultiply $\tilde{C}$ with the direct sum of the L_s, and again permute rows and columns to obtain

$$\tilde{C} \sim \begin{bmatrix} \overline{C} & 0 \\ 0 & \overline{\Delta}^2 \end{bmatrix}. \tag{1}$$

Here $\overline{C}$ is of order r and has elements

$$\overline{c}_{st} = \begin{cases} \|\alpha_s\|^2 + \delta_s^2 & \text{for all } 1 \leq j \leq r, \\ \|\alpha_s\|\|\alpha_t\| & \text{for all } 1 \leq s \neq t \leq r. \end{cases}$$

Moreover $\overline{\Delta}^2$ is diagonal, with r diagonal blocks, where block s of order $k_s - 1$ has all diagonal elements equal to δ_s^2. Thus

$$C \sim \begin{bmatrix} \overline{C} & 0 & 0 & 0 \\ 0 & \overline{\Delta}^2 & 0 & 0 \\ 0 & 0 & \tilde{\Delta}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and because $\overline{C}$ is a regular Spearman matrix this completes the proof. □

3.2. **Determinant.** Because of Theorem 3.1 it clearly suffices to compute the determinant of a regular Spearman matrix.

Theorem 3.2. *Regular Spearman matrices are non-singular (and thus positive definite).*

Proof. This is trivial for complete Spearman matrices. An incomplete Spearman matrix C can be written as

$$C = \begin{bmatrix} \Delta_{\overline{m}}^2 + \alpha_{\overline{m}}\alpha_{\overline{m}}' & \alpha_m\alpha_{\overline{m}} \\ \alpha_m\alpha_{\overline{m}}' & \alpha_m^2 \end{bmatrix},$$

where $\Delta_{\overline{m}}^2$ is Δ^2 with its last row and column deleted, and $\alpha_{\overline{m}}$ is α with its last element deleted. If $Cx = 0$ then we can suppose without loss of generality that $x_m = 1$ (we cannot have $x_m = 0$ because the leading submatrix of c is complete and thus nonsingular). Now

$$x'Cx = x_{\underline{m}}'\Delta_{\underline{m}}^2 x_{\underline{m}} + (x_{\underline{m}}'\alpha_{\underline{m}} + \alpha_m)^2.$$

The first term can only be zero if $x_{\underline{m}} = 0$, but then the second term is non-zero, because α_m is non-zero. □

Theorem 3.3. *A complete Spearman matrix $C = \alpha\alpha' + \Delta^2$ has determinant*

$$\mathbf{det}(C) = \mathbf{det}(\Delta^2)(1 + \alpha'\Delta^{-2}\alpha).$$

An incomplete Spearman matrix has determinant

$$\mathbf{det}(C) = \alpha_m^2\mathbf{det}(\Delta_{\overline{m}}^2).$$

Proof. For a complete Spearman matrix, we twice apply the classical theorem **??** on partitioned determinants (or Schur complements) to the matrix

$$D = \begin{bmatrix} \Delta^2 & \alpha \\ -\alpha' & 1 \end{bmatrix}.$$

This gives $\mathbf{det}(D) = \mathbf{det}(\Delta^2)(1+\alpha'\Delta^{-2}\alpha) = \mathbf{det}(1)\mathbf{det}(\Delta^2+\alpha\alpha')$, which proves the first part.

The second part follows by taking the limit if $\delta_m^2 \rightarrow 0$. Using the notation in the proof of Theorem 3.2,

$$\mathbf{det}(C) = \delta_m^2 \mathbf{det}(\Delta_{\overline{m}}^2)(1 + \alpha_{\overline{m}}' \Delta_{\overline{m}}^{-2} \alpha_{\overline{m}} + \frac{\alpha_m^2}{\delta_m^2}) =$$
$$= \mathbf{det}(\Delta_{\overline{m}}^2)(\delta_m^2(1 + \alpha_{\overline{m}}' \Delta_{\overline{m}}^{-2} \alpha_{\overline{m}}) + \alpha_m^2),$$

which obviously has the limit in the theorem.

Alternatively, we can use the partitioning used in the proof Theorem 3.2 and apply the partitioned determinant result to show that

$$\mathbf{det}(\Delta^2 + \alpha\alpha') = \alpha_m^2 \mathbf{det}(\Delta_{\overline{m}}^2 + \alpha_{\overline{m}} \alpha_{\overline{m}}' - \frac{\alpha_m \alpha_{\overline{m}} \alpha_m \alpha_{\overline{m}}'}{\alpha_m^2}) = \alpha_m^2 \mathbf{det}(\Delta_{\overline{m}}^2).$$

□

3.3. **Inverse.**

3.4. **Eigenvalues.** HH

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