CONVERGENCE SPEED OF BLOCK RELAXATION ALGORITHMS WITH EQUALITY RESTRICTIONS

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1. Introduction

Let us thus consider the following general situation. We minimize a real-valued twice continously differentiable function ψ defined on the product set $\Omega = \Omega_1 \otimes \Omega_2 \otimes \cdots \otimes \Omega_p$. We assume the sets Ω_s are defined by equality restrictions, in the sense that

$$\Omega_s = \{x \in \mathbb{R}^{n_s} \mid F_s(x) = 0\}$$

where the $F_s : \mathbb{R}^{n_s} \rightarrow \mathbb{R}^{m_s}$ are also assumed twice continously differentiable.

In order to minimize ψ over Ω we use the following iterative algorithm.

Date: February 16, 2004.

2000 *Mathematics Subject Classification.* 62H25.

Key words and phrases. Multivariate Analysis, Correspondence Analysis.

[Starter]	Start with $\omega^{(0)} \in \Omega$.
[Step k;Substep 1]	$\omega_1^{(k+1)} = \mathop{\mathbf{argmin}}_{\omega_1 \in \Omega_1} \psi(\omega_1, \omega_2^{(k)}, \cdots, \omega_p^{(k)})$.
[Step k;Substep 2]	$\omega_2^{(k+1)} = \mathop{\mathbf{argmin}}_{\omega_2 \in \Omega_2} \psi(\omega_1^{(k+1)}, \omega_2, \omega_3^{(k)}, \cdots, \omega_p^{(k)})$.
$\vdots$	$\vdots$
[Step k;Substep p]	$\omega_p^{(k+1)} = \mathop{\mathbf{argmin}}_{\omega_p \in \Omega_p} \psi(\omega_1^{(k+1)}, \cdots, \omega_{p-1}^{(k+1)}, \omega_p)$.
[Motor]	$k \leftarrow k+1$ and go to $k.1$

TABLE 1. Block Relaxation

2. INTRODUCTION

The following system of equations implicitly defines the updates $\hat{\omega}_s$ and the Lagrange multipliers λ_s as a function of ω_s.

$$\mathcal{D}\psi_s(\hat{\omega}_1, \cdots \hat{\omega}_s, \omega_{s+1}, \cdots, \omega_p) + \lambda_s(\hat{\omega}_1, \cdots \hat{\omega}_s, \omega_{s+1}, \cdots, \omega_p)' DF_s(\hat{\omega}_s) = 0,$$

$$F_s(\hat{\omega}_s) = 0,$$

Observe that these are $\sum_{s=1}^p (n_s + m_s)$ equations, in the same number of unknowns.

$$\mathcal{D}_t \hat{\omega}_s =$$

The subproblem updating the first set of variables x computes

(1) $$x(y) = \mathop{\mathbf{argmin}}_x \{\phi(x, y) \mid F(x) = 0\}.$$

The stationary equations for problem (1) are

$$\mathcal{D}_1\phi(x(y),y)+\mathcal{D}F(x(y))'\lambda(y)=0, \tag{2a}$$

$$F(x(y))=0. \tag{2b}$$

If we define

$$E=\mathcal{D}_{12}\phi(x(y),y), \tag{3a}$$

$$H=\mathcal{D}_{11}\phi(x(y),y)+\sum_{s-1}^{p}\lambda_s(y)\mathcal{D}^2 f_s(x(y)), \tag{3b}$$

$$G=\mathcal{D}F(x(y)). \tag{3c}$$

then we obtain, by differentiating (2),

$$\begin{bmatrix} H & G' \\ G & 0 \end{bmatrix}\begin{bmatrix} \mathcal{D}x(y) \\ \mathcal{D}\lambda(y) \end{bmatrix}=\begin{bmatrix} -E \\ 0 \end{bmatrix},$$

or

$$\mathcal{D}\lambda(y)=-(GH^{-1}G')^{-1}GH^{-1}E, \tag{4a}$$

$$\mathcal{D}x(y)=-(H^{-1}-H^{-1}G(GH^{-1}G')^{-1}GH^{-1})E. \tag{4b}$$

3. Extensions

3.1. **More than two sets.**

3.2. **Coupling constraints.**

3.3. **Inequality restrictions.** Ddodo

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