NESTED MDS SOLUTIONS

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Suppose Δ is a given matrix of dissimilarities of order n. Define $\mathcal{S}_p$ as the set of stationary points of stress $\sigma(X)$ as we vary X over all $n\times p$ configurations. Suppose $\odot$ is an $n \times r$ matrix with zeroes.

Theorem 0.1. *If $X \in \mathcal{S}_p$ then $(X \mid \odot) \in \mathcal{S}_{p+r}$*

As a corollary, the full dimensional scaling problem on configuration space is non-convex and has large numbers of non-optimal stationary points.

Suppose $Z_1, \cdots, Z_s$ are a basis and $X = \theta_1 Z_1 + \cdots + \theta_s Z_s$. Then

$$d_{ij}^2(Z) = \sum_{r=1}^{s}\sum_{t=1}^{s}\theta_r\theta_t \mathbf{tr}\, Z_r' A_{ij} Z_t$$

which we can write as $d_{ij}^2(\theta) = \theta' C_{ij}\theta$. Thus

$$\sigma(\theta) = 1 + \frac{1}{2}\theta' V\theta - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\delta_{ij}\sqrt{\theta' C_{ij}\theta},$$

with

$$V = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} C_{ij}$$

The matrices C_{ij} are of order s. Suppose T is such that $T'VT = I$ and define $\xi = T^{-1}\theta$. Then $\theta' V\theta = \xi'\xi$, and

$$\sigma(\xi) = 1 + \frac{1}{2}\xi'\xi - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\delta_{ij}\sqrt{\xi' U_{ij}\xi},$$

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where $U_{ij} = T'C_{ij}T$, which implies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}U_{ij} = I$$

The stationary equations at a point where stress is differentiable are simply $B(\xi)\xi = \xi$, where

$$B(\xi) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}(\xi)}U_{ij}$$

Now suppose the Z_r can be chosen in such a way that all U_{ij} are direct sums of p matrices of orders $s_1, \cdots, s_p$. Then the stationary equations look like $B_r(\xi)\xi_r = \xi_r$. We can find solutions to these stationary equations by setting some of the ξ_r equal to zero. Thus MDS solutions are nested in the sense that solutions for dimensionality p are also solutions for dimensionality $q > p$.

The second derivatives are

$$I - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}(\xi)}\left[U_{ij} - \frac{U_{ij}\xi\xi' U_{ij}}{\xi' U_{ij}\xi}\right]$$

This happens, for example, if the Z_r are the np unit coordinate matrices of the form e_ie_q' in which case the C_{ij} are direct sums of p matrices of order n.

Related result: suppose we require some distances to be zero. Then the solution to that constrained problem also solves the stationary equations for the unconstrained problem.

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