PERTURBATION OF ORTHOGONALIZATIONS

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1. Introduction

Suppose X is a given $n \times m$ matrix with $n \geq m$ and $\mathbf{rank}(X) = m$. Suppose Z is an *orthogonalization* of X, i.e. $Z = XS$ with $Z'Z = I$. We also write this as $Z = \mathbf{orth}(X)$. If $Z_1 = XS_1$ and $Z_2 = XS_2$ are two orthogonalizations of X, then there is a square orthonormal (rotation) matrix R such that $Z_1 = Z_2R$ and $S_2 = S_1R'$.

Three particular choices for orthogonalization obtained from QR-decomposition, in which S is upper-triangular, from LS-decomposition, in which S is symmetric, and from SV-decomposition in which S is orthogonal. In all three cases we want to study the effect of small perturbation of X on Z and S.

2. Perturbation Equations

Perturbing X to $X + \Delta$ will perturb Z to $Z + \Xi$ and S to $S + \Sigma$. Equating first order terms on both sides of $[X + \Delta][S + \Sigma] = Z + \Xi$ gives

$$\Delta S + X\Sigma = \Xi. \tag{1a}$$

Equating first order terms on both sides of $[Z + \Xi]'[Z + \Xi] = I$ gives

$$Z'\Xi + \Xi' Z = 0. \tag{1b}$$

Let $M \triangleq Z'\Delta$ and $N \triangleq Z'X$. Then, from $Z = XS$, we see that $I = NS$ and thus $N = S^{-1}$. We also see

$$MS + N\Sigma = Z'\Xi,$$

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and thus

$$N\Sigma + \Sigma' N' = -(MS + S'M'). \tag{2}$$

Define $U \stackrel{\Delta}{=} -(MS + S'M')$.

3. Triangular Case

If S is upper-triangular, then N and Σ are upper triangular too. Suppose **upp**() is the upper-triangular part of a matrix (including the diagonal) From Equation (2)

$$\begin{aligned} N\Sigma + \mathbf{diag}(N\Sigma) &= \mathbf{upp}(U), \\ 2\mathbf{diag}(N\Sigma) &= \mathbf{diag}(U). \end{aligned}$$

and thus

$$N\Sigma = \mathbf{upp}(U) - \frac{1}{2}\mathbf{diag}(U).$$

We use V for the upper-triangular matrix on the right hand side of this equation. Clearly $V + V' = U$, and

$$\Sigma = N^{-1}V = SV.$$

This implies

$$\Xi = \Delta S + ZV$$

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4. Symmetric Case

If S is symmetric, then N and Σ are symmetric too. Suppose $X = K\Lambda L'$ is the singular value decomposition of X. Then $S = L\Lambda^{-1}L'$ and $N = S^{-1} = L\Lambda L'$. Define $\Gamma \stackrel{\Delta}{=} L'\Sigma L$ and $\Phi \stackrel{\Delta}{=} K'\Delta L$. Equation (2) gives

$$\Lambda\Gamma + \Gamma\Lambda = -[\Phi\Lambda^{-1} + \Lambda^{-1}\Phi'],$$

and thus

$$\gamma_{ij} = -\frac{\lambda_i\phi_{ij} + \lambda_j\phi_{ji}}{\lambda_i\lambda_j(\lambda_i + \lambda_j)}$$

Now define $A \stackrel{\Delta}{=} K'\Xi L$ and $B \stackrel{\Delta}{=} K_\perp'\Xi L$, where $K_\perp$ is an orthonormal basis for the complement of K. From Equation (1a)

$$a_{ij} = \frac{\phi_{ij} - \phi_{ji}}{\lambda_i + \lambda_j},$$

and

$$b_{ij} = \frac{\psi_{ij}}{\lambda_j},$$

where $\Psi \stackrel{\Delta}{=} K_\perp'\Delta L$. Now $\Xi = KAL' + K_\perp BL'$, and thus

$$\xi_{pq} = \sum_{i=1}^m \sum_{j=1}^m \left[\frac{\phi_{ij} - \phi_{ji}}{\lambda_i + \lambda_j}\right] k_{pi} l_{qj} + \sum_{i=1}^{n-m} \sum_{j=1}^m \left[\frac{\psi_{ij}}{\lambda_j}\right] k_{pi}^\perp l_{qj}. \tag{3a}$$

Also $\Sigma = L\Gamma L'$ and thus

$$\sigma_{pq} = -\sum_{i=1}^m \sum_{j=1}^m \left[\frac{\lambda_i\phi_{ij} + \lambda_j\phi_{ji}}{\lambda_i\lambda_j(\lambda_i + \lambda_j)}\right] l_{pi} l_{pj}. \tag{3b}$$

5. Orthogonal Case

6. Partial Derivatives

7. Rank Deficiency

8. Application to Simultaneous Power Iterations

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