THE LUB OF PSD MATRICES

JAN DE LEEUW

1. Introduction

Suppose $\mathcal{A}_n$ is the space all of real symmetric matrices of order n, and $\mathcal{P}_n \subset \mathcal{A}_n$ is the cone of positive semi-definite matrices in that space. For any two $A_1, A_2 \in \mathcal{A}_n$ we say that $A_1 \gtrsim A_2$ if $A_1 - A_2 \in \mathcal{P}_n$. This defines the *Loewner partial order* on $\mathcal{A}_n$.

Theorem 1.1. *The poset $(\mathcal{P}_n, \gtrsim)$ is a lattice.*

Proof. We need to show that all pairs $P_1, P_2 \in \mathcal{P}$ have a least upper bound (or join) and a greatest lower bound (or meet). For any pair of positive semidefinite matrices there is a nonsingular S such that $P_1 = S\Lambda_1 S'$ and $P_2 = S\Lambda_2 S'$, with Λ_1 and Λ_2 diagonal. Now define

$$P_1 \cup P_2 = S\ \max(\Lambda_1, \Lambda_2)S'.$$

Clearly $P_1 \cup P_2 \gtrsim P_1$ and $P_1 \cup P_2 \gtrsim P_2$, and thus $P_1 \cup P_2$ is an upper bound.

Let us assume the columns of S are ordered in such a way that the first n_+ elements of $\Lambda_1 - \Lambda_2$ along the diagonal are positive, the next n_0 elements are zero, and the last n_- elements are negative. We partition S in the same way into three submatrices S_+, S_0, and S_-.

Thus

$$P_1 \cup P_2 - P_2 = S(\max(\Lambda_1, \Lambda_2) - \Lambda_2)S' = S_+(\Lambda_1 - \Lambda_2)_+ S_+',$$
$$P_1 \cup P_2 - P_1 = S(\max(\Lambda_1, \Lambda_2) - \Lambda_1)S' = S_-(\Lambda_2 - \Lambda_1)_- S_-'.$$

Date: January 2, 2006.

2000 *Mathematics Subject Classification.* 62H25.

Key words and phrases. Multivariate Analysis, Correspondence Analysis.

Suppose $P \gtrsim P_1$ and $P \gtrsim P_2$ and $P_1 \cup P_2 \gtrsim P$. Thus P is a smaller upper bound than $P_1 \cup P_2$. We see from

$$P_1 \cup P_2 - P_1 \gtrsim P_1 \cup P_2 - P,$$
$$P_1 \cup P_2 - P_2 \gtrsim P_1 \cup P_2 - P,$$

that the null-space of $P_1 \cup P_2 - P$ must contain both the null-space of $P_1 \cup P_2 - P_2$ and the null-space of $P_1 \cup P_2 - P_1$.

Now S_+ is $n \times n_+$ of rank n_+. Thus there exists an $n \times n - n_+$ matrix T_+ of rank $n - n_+$ such that $T_+'S_+ = 0$. □

It follows that if $P_1, \cdots, P_m \in \mathcal{P}_n$, then the least upper bound $P_1 \cup \cdots \cup P_m$ exists. Moreover it can be computed by recursively applying the associative law

$$P_1 \cup \cdots \cup P_m = (P_1 \cup \cdots \cup P_{m-1}) \cup P_m.$$

Department of Statistics, University of California, Los Angeles, CA 90095-1554

E-mail address, Jan de Leeuw: deleeuw@stat.ucla.edu

URL, Jan de Leeuw: http://gifi.stat.ucla.edu