THE NEGATIVE LOGARITHM OF THE CUMULATIVE NORMAL

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ABSTRACT. In this note we discuss the probit function, for which the best quadratic majorization is the uniform quadratic majorization given by an upper bound for the second derivative.

1. DEFINITION

We define the *normal density*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2),$$

and the *normal distribution function*

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$$

in the usual way. In addition we define

$$f(x) = -\log \Phi(x).$$

2. DERIVATIVES

Clearly

$$f'(x) = -\frac{\phi(x)}{\Phi(x)}$$

$$f''(x) = \frac{x\phi(x)}{\Phi(x)} + \left[\frac{\phi(x)}{\Phi(x)}\right]^2.$$

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We can get more insight into these derivatives by rewriting them as conditional expectations. If $u = \phi(z)$ then $du = -z\phi(z)dz$ and thus

$$\int_{-\infty}^{x} z\phi(z)dz = -\int_{0}^{\phi(x)} du = -\phi(x),$$

which implies

$$f'(x) = \frac{\int_{-\infty}^{x} z\phi(z)dz}{\int_{-\infty}^{x} \phi(z)dz} = \mathbf{E}(z|z < x).$$

This shows that $f'(x) < 0$ and thus f is decreasing.

Now in the same way we can define $u = z\phi(z)$ and use $du = (1 - z^2)\phi(z)$ to derive

$$\int_{-\infty}^{x} (1 - z^2)\phi(z)dz = \int_{0}^{x\phi(x)} du = x\phi(x),$$

which implies

$$1 - \mathbf{E}(z^2|z < x) = \frac{x\phi(x)}{\Phi(x)},$$

and thus

$$f''(x) = 1 - [\mathbf{E}(z^2|z < x) + \mathbf{E}(z|z < x)] = 1 - \mathbf{V}(z|z < x).$$

This shows that $0 < f''(x) < 1$, and thus f is convex and has a bounded second derivative. Moreover $f''(x)$ is decreasing, which implies that f' is concave. Also

$$\lim_{x \to -\infty} f''(x) = 1,$$
$$\lim_{x \to +\infty} f''(x) = 0.$$

3. Quadratic Majorization

A function g majorizes our function f in a point y if $g(x) \geq f(x)$ for all x and $g(y) = f(y)$. A quadratic function

$$g(x) = c + b(x - y) + \frac{1}{2}a(x - y)^2$$

majorizes f in y if and only if $c = f(y)$, $b = f'(y)$, and

$$a \geq A(y) = \sup_{x \neq y} \delta(x|y),$$

where

$$\delta(x|y) = \frac{f(x) - f(y) - f'(y)(x - y)}{\frac{1}{2}(x - y)^2}.$$

We find the *best quadratic majorization* of f in y by choosing $a = A(y)$.

Since $\delta(x|y = f''(z)$ for some z between x and y, we see that $\delta(x|y) < 1$ for all x. On the other hand

$$\lim_{x\rightarrow-\infty} \delta(x, y) = 1,$$

and consequently $A(y) = 1$ for all y. Thus the best quadratic majorization is actually the *uniform quadratic majorization*

$$g(x) = f(y) + f'(y)(x - y) + \frac{1}{2}(x - y)^2.$$

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