DOUBLE CENTERING PERTURBATIONS

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ABSTRACT. Meet the abstract. This is the abstract.

1. INTRODUCTION

Suppose A is a square symmetric matrix of order n, with non-negative elements, and spectral decomposition $A = X\Lambda X'$. Eigenvalues are non-increasing along the diagonal. Suppose e_n is a vector with all n elements equal to one, and J_n is the centering operator $J_n = I_n - \frac{1}{n}e_ne_n'$. Define $B = J_nAJ_n$. We want to relate the spectral decomposition of $B = Y\Omega Y'$ to that of A.

2. LOW-RANK PERTURBATION

Write the double centered matrix B as

$$B = A - e_na' - ae_n' + \alpha e_ne_n',$$

where $a = n^{-1}Ae_n$ and $\alpha = n^{-2}e_n'Ae_n$. We suppose throughout that $\alpha > 0$. Alternatively

$$B = A + \alpha(e_n - \frac{a}{\alpha})(e_n - \frac{a}{\alpha})' - \alpha^{-1}aa',$$

showing that B is a rank-two perturbation of A. If we define the $n \times 2$ matrix

$$H = \begin{bmatrix}\alpha^{\frac{1}{2}}(e_n - \frac{a}{\alpha}) & \alpha^{-\frac{1}{2}}a\end{bmatrix}$$

and the 2×2 diagonal matrix

$$D = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

then $B = A - HDH'$.

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We now use the Schur determinant theorem (see, for example, De Leeuw [1986]) twice on the matrix

$$\begin{bmatrix} A-\mu I & H \\ H' & D \end{bmatrix}$$

If $A-\mu I$ is non-singular, then

$$\mathbf{det}(B-\mu I) = -\mathbf{det}(A-\lambda I)\mathbf{det}(D-H'(A-\mu I)^{-1}H).$$

Now remember that $A = K\Lambda K'$. Define u as the first column and v as the second column of $K'H$. Then

$$\mathbf{det}(D-H'(A-\mu I)^{-1}H) = \mathbf{det}\begin{bmatrix} -1-\sum_{i=1}^n \frac{u_i^2}{\lambda_i-\mu} & -\sum_{i=1}^n \frac{u_iv_i}{\lambda_i-\mu} \\ -\sum_{i=1}^n \frac{u_iv_i}{\lambda_i-\mu} & 1-\sum_{i=1}^n \frac{v_i^2}{\lambda_i-\mu} \end{bmatrix}$$

which means we must solve

$$G(\mu) \triangleq (1+\sum_{i=1}^n \frac{u_i^2}{\lambda_i-\mu})(1-\sum_{i=1}^n \frac{v_i^2}{\lambda_i-\mu}) + \left(\sum_{i=1}^n \frac{u_iv_i}{\lambda_i-\mu}\right)^2 = 0,$$

or, perhaps more conveniently,

$$F(\mu) \triangleq \frac{(1+\sum_{i=1}^n \frac{u_i^2}{\lambda_i-\mu})(1-\sum_{i=1}^n \frac{v_i^2}{\lambda_i-\mu})}{\left(\sum_{i=1}^n \frac{u_iv_i}{\lambda_i-\mu}\right)^2} = -1.$$

Note that if all λ_i are different, then

$$\lim_{\mu\rightarrow\lambda_i} F(\mu) = -1.$$

Of course if ω is an eigenvalue of B, unequal to one of the eigenvalues of A, we have $F(\omega) = 1$ as well.

3. Interlacing

The centering operator J_n is an orthogonal projector of rank $n-1$, and thus there is an orthonormal $n\times(n-1)$ matrix M such that $J_n = MM'$. It follows that the non-zero eigenvalues of B are the eigenvalues of $M'AM$.

4. Expansion

Partition X and Λ, singling out the Perron-Frobenius eigenvector and eigenvalue, as

$$X = \begin{bmatrix} x & \overline{X} \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\Lambda} \end{bmatrix}.$$

Define $h = n^{-\frac{1}{2}}X'e$. Note that $h'h = 1$, which means we can write h in the form

$$h = \begin{bmatrix} \sqrt{1-\epsilon^2} \\ \epsilon u \end{bmatrix}$$

for some u with $u'u = 1$ and some $0 \leq \epsilon \leq 1$. Also define $v = \overline{\Lambda}u$ and $\kappa = u'\overline{\Lambda}u$. It follows from the definitions so far that $\alpha = \frac{1}{n}((1-\epsilon^2)\lambda + \epsilon^2\kappa)$.

If $C = X'BX$, then

$$C = \Lambda - hh'\Lambda - \Lambda hh' + ((1-\epsilon^2)\lambda + \epsilon^2\kappa)hh'.$$

Of course C has the same eigenvalues as B, and if $C = Z\Omega Z'$ then $B = (XZ)\Omega(XZ)'$.

We start the comptations by using

$$hh' = \begin{bmatrix} 1-\epsilon^2 & \epsilon\sqrt{1-\epsilon^2}u' \\ \epsilon\sqrt{1-\epsilon^2}u & \epsilon^2 uu' \end{bmatrix}.$$

We can then write C as

$$\begin{bmatrix} \epsilon^2\kappa + \epsilon^4(\lambda-\kappa) & -\epsilon\sqrt{1-\epsilon^2}(v + \epsilon^2(\lambda-\kappa)u)' \\ -\epsilon\sqrt{1-\epsilon^2}(v + \epsilon^2(\lambda-\kappa)u) & \Lambda - \epsilon^2(uv' + vu' - \lambda uu') - \epsilon^4(\lambda-\kappa)uu' \end{bmatrix}$$

So far, these results are exact, and they do not depend in any way on ϵ being small.

We can turn the exact result into a convergent power series for C by using $\sqrt{1-\epsilon^2} = 1 - \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 - \cdots$. This allows us to write

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots$$

We find

$$C_0 = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & -v' \\ -v & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} \kappa & 0 \\ 0 & -(uv' + vu' - \lambda uu') \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & \frac{1}{2}v - (\lambda - \kappa)u' \\ \frac{1}{2}v - (\lambda - \kappa)u & 0 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} \lambda - \kappa & 0 \\ 0 & -(\lambda - \kappa)uu' \end{bmatrix},$$

and so on. Observe the diagonal submatrices only have even powers, the off-diagonal submatrices only have odd powers. Of course the series will converge fast if ϵ is small.

In any case, a convergent power series for a matrix can be turned into a convergent power series for its eigenvalues and eigenvectors. This can be used for perturbation analysis, most easily for the eigenvalues [Kato, 1976; Baumgärtel, 1985].

5. sin Θ theorems

Alternatively, for the eigenvectors the $\sin\Theta$ theorems may be the more natural way to go [Ipsen, 2000]. A relevant recent contribution is Chen and Li [2006].

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