THE SPEARMAN MODEL

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1. Spearman Correlations

Definition 1.1. A *correlation function* or CF on a set T is a function $\rho : T \otimes T \Rightarrow \mathbb{R}$ such that

- $\rho(t,t) = 1$ for all $t \in T$,
- $\rho(s,t) = \rho(t,s)$ for all $s,t \in T$,
- If $\{t_1, \cdots, t_n\}$ is a finite set of elements of T, then the $n \times n$ matrix with elements $\rho_{ij} = \rho(t_i, t_j)$ is positive semi-definite.

Definition 1.2. A correlation ρ is *a Spearman correlation function* or SCF if there exist $\alpha : T \Rightarrow \mathbb{R}$ with $-1 \leq \alpha(t) \leq +1$ for all $t \in T$ such that $\rho(s,t) = \alpha(s)\alpha(t)$ for all $s \neq t$. An SCF is *proper* or a PSCF if $0 < \alpha(t) < 1$ for all $t \in T$.

Definition 1.3. We call $\alpha^2(t)$ the *common variance* at t and $\omega^2(t) \triangleq 1 - \alpha^2(t)$ the *unique variance* at t.

Theorem 1.1. *If ρ is an SFT then there is a partition of T into subsets T_0, T_1, and $T_\star$ such that*

- *If $t \in T_0$ then $\rho(t,s) = 1$ for $t = s$ and $\rho(t,s) = 0$ otherwise.*
- *If $t \in T_1$ then $\rho(t,s) = 1$ for $s \in T_1$ and $\rho(t,s) = 0$ otherwise.*
- *If $t \in T_+$ then $\rho(t,s) \neq 0$ for all $s \in T$.*

Proof. The three blocks correspond with $\alpha(t) = 0$, $\alpha(t) = 1$, and $-1 < \alpha(t) < +1$. □

Thus ρ is the direct sum of three blocks, where block on T_0 ρ is the identity, on block T_1 it is identically one, and on block T_+ it is non-zero

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everywhere. By allowing for sign changes, we can choose ρ on block T_+ to be positive, which means it is a PSCF on T_+.

Theorem 1.2. *A correlation ρ on T is a PSCF if and only if*

(1) $\rho(s,t) > 0$ *for all* $s,t \in T$,
(2) $\rho(s,t)\rho(u,v) = \rho(s,v)\rho(t,u)$ *for all quadruples* (s,t,u,v) *of distinct elements of* T,
(3) $\rho(t,u) > \rho(t,s)\rho(s,u)$ *for all triples* (s,t,u) *of distinct elements of* T.

Proof. The proof is exactly the same as the one in Bekker and De Leeuw [1987] for the case in which T is finite. The 2×2 matrices

$$\begin{bmatrix} \rho(s,u) & \rho(s,v) \\ \rho(t,u) & \rho(t,v) \end{bmatrix}$$

must be of rank one. This is condition (2). We set

$$\hat{\alpha}(s,u,v) \stackrel{\Delta}{=} \sqrt{\frac{\rho(s,u)\rho(s,v)}{\rho(u,v)}}.$$

Then condition (2) shows that the value of $\hat{\alpha}(s) \stackrel{\Delta}{=} \hat{\alpha}(s,u,v)$ is independent of u and v and that $\rho(s,t) = \hat{\alpha}(s)\hat{\alpha}(t)$. Condition (3) comes from the fact that the 2×2 matrices

$$\begin{bmatrix} 1 & \rho(s,u) \\ \rho(t,s) & \rho(t,u) \end{bmatrix}$$

must be positive definite. It guarantees that $\hat{\alpha}^2(s) < 1$. □

2. Linear Spearman Structures

Suppose $\mathcal{Y}$ is a separable pre-Hilbert space with inner product $< \bullet, \bullet >$ and with unit sphere $S \stackrel{\Delta}{=} \{y \in \mathcal{Y} \mid < y, y >= 1\}$.

Definition 2.1. A function y on a set T with values in S is a *Linear Spearman Structure* or LSS if there exist

- an element $u \in S$,
- a function e on T with values in S,
- a real-valued function α on T, with $-1 \leq \alpha(t) \leq +1$ for all $t \in T$,

such that

- $y(t) = \alpha(t)u + \omega(t)e(t)$ for all $t \in T$,
- $< u, e(t) >= 0$ for all $t \in T$,
- $< e(s), e(t) >= 0$ for all $s \neq t \in T$.

The LSS is proper, or a PLSS, if $-1 \leq \alpha(t) \leq +1$ for all $t \in T$.

As before, we use the shorthand $\omega(t)$ñ $\stackrel{\Delta}{=} \sqrt{1 - \alpha^2(t)}$.

Theorem 2.1. *If y is an LSS, then the CF defined by $\rho(s,t) =< y(s), y(t) >$ is an SCF.*

Proof. Definition 2.1 gives $< y(s), y(t) >= \alpha(s)\alpha(t) + \omega^2(t)\delta^{st}$. □

Note that in addition in an LSS

$$< y(t), u > = \alpha(t) \text{ for all } t \in T, \tag{1a}$$

$$< y(t), e(s) > = \omega(t)\delta^{st}. \tag{1b}$$

$$y(t) = \sum_{i=0}^{\infty} \eta_i(t) z_i,$$
$$e(t) = \sum_{i=0}^{\infty} \epsilon_i(t) z_i,$$
$$u = \sum_{i=0}^{\infty} \mu_i z_i.$$

$$\eta_i(t) = \alpha(t)\mu_i + \omega(t)\epsilon_i(t)$$

Suppose $[Y]$ is the span of y, i.e. the set of all finite linear combinations $\sum_{i=1}^n \beta_i y(t_i)$. Moreover $\overline{[Y]}$ is the closure of the span. Also $[Y]_\perp$ is the perp of y, i.e. the closed linear subspace of all $z \in \mathcal{Y}$ such that $< z, y(t) >= 0$ for all $t \in T$.

3. Fundamental Theorem of Factor Analysis

3.1. Existence.

Theorem 3.1. *If $\{y_j \in \mathcal{Y}\}_{1\leq j\leq m}$ is a Spearman sequence, then C with elements $c_{j\ell} =< y_j, y_\ell >$ is a Spearman matrix. Conversely, if C is a Spearman matrix, then there exists a Spearman sequence $\{y_j \in \mathcal{Y}\}_{1\leq j\leq m}$ with $c_{j\ell} =< y_j, y_\ell >$.*

Proof. The first part is a simple calculation. □

3.2. Indeterminacy.

4. Properties of Spearman Matrices

4.1. Canonical Form.

Definition 4.1. A Spearman matrix is *regular* if

- $\alpha_j > 0$ for all $1 \leq j \leq m$,
- $\delta_1^2 > \delta_2^2 > \cdots > \delta_m^2$.

A regular Spearman matrix is *complete* if $\delta_m^2 > 0$, otherwise it is *incomplete*.

Theorem 4.1. *Each Spearman matrix is (orthogonally) similar to the direct sum of a regular Spearman matrix and a diagonal matrix.*

Proof. We first permute rows and columns of the Spearman matrix C such that those with both $\alpha_j = 0$ and $\delta_j^2 = 0$ come last. Suppose there are m_{00} of these.

Then permute again so that the, say, m_{01} rows and columns with $\alpha_j = 0$ and $\delta_j^2 > 0$ come before these. We then have

$$C \sim \begin{bmatrix} \tilde{C} & 0 & 0 \\ 0 & \tilde{\Delta}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now look at the submatrix $\tilde{C}$, for which all $\alpha_j \neq 0$. Permute again to make the δ_j^2 non-increasing along the diagonal. Suppose the δ_j^2 have r

different values, and that value δ_s^2 has multiplicity k_s. Write α_s for the subvector of α corresponding to δ_s^2.

Construct the r orthonormal matrices L_s, of order k_s, whose first columns are equal to $\alpha_s/\|\alpha_s\|$, and whose remaining columns are orthogonal to α_s, and to each other. Premultiply $\tilde{C}$ with the direct sum of the L_s, and again permute rows and columns to obtain

$$\tilde{C} \sim \begin{bmatrix} \overline{C} & 0 \\ 0 & \overline{\Delta}^2 \end{bmatrix}. \tag{2}$$

Here $\overline{C}$ is of order r and has elements

$$\overline{c}_{st} = \begin{cases} \|\alpha_s\|^2 + \delta_s^2 & \text{for all } 1 \le j \le r, \\ \|\alpha_s\|\|\alpha_t\| & \text{for all } 1 \le s \ne t \le r. \end{cases}$$

Moreover $\overline{\Delta}^2$ is diagonal, with r diagonal blocks, where block s of order $k_s - 1$ has all diagonal elements equal to δ_s^2. Thus

$$C \sim \begin{bmatrix} \overline{C} & 0 & 0 & 0 \\ 0 & \overline{\Delta}^2 & 0 & 0 \\ 0 & 0 & \tilde{\Delta}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and because $\overline{C}$ is a regular Spearman matrix this completes the proof. □

4.2. **Determinant.** Because of Theorem 4.1 it clearly suffices to compute the determinant of a regular Spearman matrix.

Theorem 4.2. *Regular Spearman matrices are non-singular (and thus positive definite).*

Proof. This is trivial for complete Spearman matrices. An incomplete Spearman matrix C can be written as

$$C = \begin{bmatrix} \Delta_{\overline{m}}^2 + \alpha_{\overline{m}}\alpha_{\overline{m}}' & \alpha_m\alpha_{\overline{m}} \\ \alpha_m\alpha_{\overline{m}}' & \alpha_m^2 \end{bmatrix},$$

where $\Delta_{\overline{m}}^2$ is Δ^2 with its last row and column deleted, and $\alpha_{\overline{m}}$ is α with its last element deleted. If $Cx = 0$ then we can suppose without loss of generality that $x_m = 1$ (we cannot have $x_m = 0$ because the leading submatrix of c is complete and thus nonsingular). Now

$$x'Cx = x'_{\underline{m}}\Delta^2_{\underline{m}}x_{\underline{m}} + (x'_{\underline{m}}\alpha_{\underline{m}} + \alpha_m)^2.$$

The first term can only be zero if $x_{\underline{m}} = 0$, but then the second term is non-zero, because α_m is non-zero. □

Theorem 4.3. *A complete Spearman matrix $C = \alpha\alpha' + \Delta^2$ has determinant*

$$\mathbf{det}(C) = \mathbf{det}(\Delta^2)(1 + \alpha'\Delta^{-2}\alpha).$$

An incomplete Spearman matrix has determinant

$$\mathbf{det}(C) = \alpha^2_m\mathbf{det}(\Delta^2_{\overline{m}}).$$

Proof. For a complete Spearman matrix, we twice apply the classical theorem **??** on partitioned determinants (or Schur complements) to the matrix

$$D = \begin{bmatrix} \Delta^2 & \alpha \\ -\alpha' & 1 \end{bmatrix}.$$

This gives $\mathbf{det}(D) = \mathbf{det}(\Delta^2)(1 + \alpha'\Delta^{-2}\alpha) = \mathbf{det}(1)\mathbf{det}(\Delta^2 + \alpha\alpha')$, which proves the first part.

The second part follows by taking the limit if $\delta^2_m \rightarrow 0$. Using the notation in the proof of Theorem 4.2,

$$\begin{aligned}\mathbf{det}(C) = \delta^2_m\mathbf{det}(\Delta^2_{\overline{m}})(1 + \alpha'_{\overline{m}}\Delta^{-2}_{\overline{m}}\alpha_{\overline{m}} + \frac{\alpha^2_m}{\delta^2_m}) = \\ = \mathbf{det}(\Delta^2_{\overline{m}})(\delta^2_m(1 + \alpha'_{\overline{m}}\Delta^{-2}_{\overline{m}}\alpha_{\overline{m}}) + \alpha^2_m),\end{aligned}$$

which obviously has the limit in the theorem.

Alternatively, we can use the partitioning used in the proof Theorem 4.2 and apply the partitioned determinant result to show that

$$\mathbf{det}(\Delta^2 + \alpha\alpha') = \alpha^2_m\mathbf{det}(\Delta^2_{\overline{m}} + \alpha_{\overline{m}}\alpha'_{\overline{m}} - \frac{\alpha_m\alpha_{\overline{m}}\alpha_m\alpha'_{\overline{m}}}{\alpha^2_m}) = \alpha^2_m\mathbf{det}(\Delta^2_{\overline{m}}).$$

□

4.3. **Inverse.**

4.4. **Eigenvalues.**

References

P. Bekker and J. De Leeuw. The Rank of Reduced Dispersion Matrices. *Psychometrika*, 52:125–135, 1987.

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