SHARP BROKEN-LINE MINORIZATION

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1. Introduction

Suppose f is a real function of a real variable. For each $x \neq y$ define

$$\delta_f(x,y) \stackrel{\Delta}{=} \frac{f(x)-f(y)}{x-y}.$$

If f is differentiable at y we set $\delta_f(y,y) = f'(y)$. Also define

$$\underline{\delta}_f(y) \stackrel{\Delta}{=} \inf_{x>y} \delta_f(x,y),$$

$$\overline{\delta}_f(y) \stackrel{\Delta}{=} \sup_{x<y} \delta_f(x,y).$$

Of course $\underline{\delta}_f(y)$ could be $-\infty$ and/or $\overline{\delta}_f(y)$ could be $+\infty$, and we will take these possibilities into account.

Note that if f is differentiable at y and $\delta_f(x,y)$ is increasing in x then $\underline{\delta}_f(y) = \overline{\delta}_f(y) = f'(y)$. If $\delta_f(x,y)$ is decreasing in x then $\underline{\delta}_f(y) = \lim_{x\rightarrow+\infty} \delta_f(x,y)$ and $\overline{\delta}_f(y) = \lim_{x\rightarrow-\infty} \delta_f(x,y)$.

If $x > y$ then $\delta_f(x,y) \geq \underline{\delta}_f(y)$ and thus

$$f(x) \geq f(y) + \underline{\delta}_f(y)(x-y).$$

If $x < y$ then $\delta_f(x,y) \leq \overline{\delta}_f(y)$ and thus also

$$f(x) \geq f(y) + \overline{\delta}_f(y)(x-y).$$

This means that if we define the extended real valued function

$$h(x,y) \stackrel{\Delta}{=} \begin{cases} f(y) + \overline{\delta}_f(y)(x-y) & \text{if } x < y, \\ f(y) + \underline{\delta}_f(y)(x-y) & \text{if } x > y, \\ f(y) & \text{if } x = y, \end{cases}$$

then $f(x) \geq h(x,y)$ for all x and y and we have a minorization function consisting of two line segments.

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2. Examples

2.1. **Absolute Value.** If $f(x) = |x|$ then $\delta_f(x,0) = \mathbf{sign}(x)$. Thus $\underline{\delta}_f(0) = 1$ and $\overline{\delta}_f(0) = -1$. It follows that $h(x,0) = |x|$, as expected.

2.2. **Quadratic.** If $f(x) = ax^2 + bx + c$ then $\delta_f(x,y) = a(x+y) + b$. Thus if $a > 0$ we have $\underline{\delta}_f(y) = \overline{\delta}_f(y) = a$ and $h(x,y) = f(y) + a(x-y)$. If $a < 0$ we have $\underline{\delta}_f(y) = -\infty$ and $\overline{\delta}_f(y) = +\infty$.

2.3. **Cubic.** If $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$, then $\delta_f(x,y) = ax^2 + (ay + b)x + (ay^2 + by + c)$. Suppose $a > 0$. Then $\overline{\delta}_f(y) = +\infty$. The minimum of $\delta_f(x,y)$ over x is attained at $-(ay+b)/2a$, and thus

$$\underline{\delta}_f(y) = \begin{cases} f'(y) & \text{if } y \geq -\frac{b}{3a}, \\ \min_x \delta_f(x,y) & \text{if } y < -\frac{b}{3a}. \end{cases}$$

If $a < 0$ we find, in the same way, that $\underline{\delta}_f(y) = -\infty$ and that

$$\overline{\delta}_f(y) = \begin{cases} f'(y) & \text{if } y \leq -\frac{b}{3a}, \\ \max_x \delta_f(x,y) & \text{if } y > -\frac{b}{3a}. \end{cases}$$

2.4. **Quartic.** Consider the quartic $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, with $a \neq 0$. We have

$$\delta_f(x,y) = ax^3 + (ay+b)x^2 + (ay^2 + by + c)x + (ay^3 + by^2 + cy + d).$$

Also the derivative of δ_f with respect to x is is the quadratic

$$\delta_f'(x,y) = 3ax^2 + 2(ay+b)x + (ay^2 + by + c).$$

First suppose $a < 0$. This case turns out to be uninteresting, because $\underline{\delta}_f(y) = -\infty$ and $\overline{\delta}_f(y) = +\infty$. So assume $a > 0$. If $\delta_f'(x,y)$ has no real roots (or two equal real roots), as a function of x for fixed y, then $\delta_f'(x,y) \geq 0$ for all x and $\delta_f(x,y)$ is increasing in x, and $\underline{\delta}_f(y) = \overline{\delta}_f(y) = f'(y)$.

If $\delta_f'(x,y)$ has two real roots, then $\delta_f(x,y)$ has a local maximum at the smallest root, say x_1, and a local minimum at the largest root, say x_2. There is also a $x_0 < x_1$

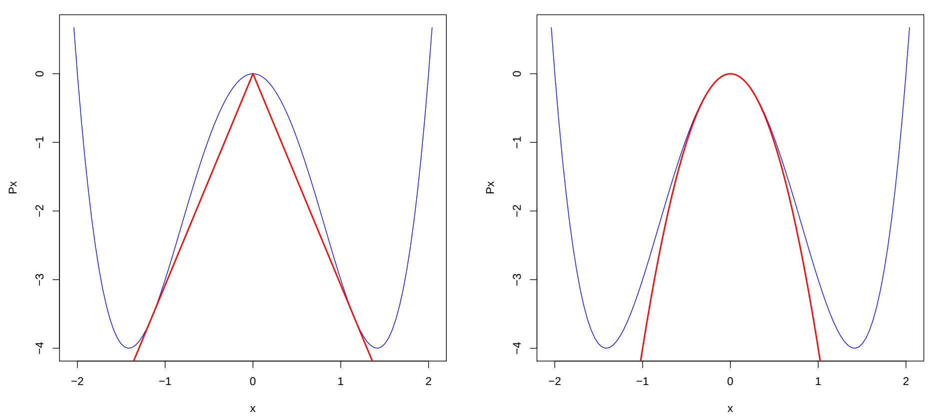
with $\delta_f(x_0,y) = \delta_f(x_2,y)$ and an $x_3 > x_2$ such that $\delta_f(x_3,y) = \delta_f(x_1,y)$. Now

$$\underline{\delta}_f(y) = \begin{cases} f'(y) & \text{if } y \geq x_0, \\ \delta_f(x_2,y) & \text{if } x_0 \leq y \leq x_2, \\ f'(y) & \text{if } y \geq x_2. \end{cases}$$

Of course in the same way

$$\overline{\delta}_f(y) = \begin{cases} f'(y) & \text{if } y \leq x_1, \\ \delta_f(x_1,y) & \text{if } x_1 \leq y \leq x_3, \\ f'(y) & \text{if } y \geq x_3. \end{cases}$$

A simple numerical example sets $a = 1$, $c = -4$, and $b = d = e = 0$. Thus $f(x) = x^4 - 4x^2$. Moreover $\delta_f(x,0) = x^3 - 4x$, and $\delta_f'(x,0) = 3x^2 - 4$. The roots of the quadratic are $x_1 = -\frac{2}{3}\sqrt{3}$ and $x_2 = +\frac{2}{3}\sqrt{3}$. Also $x_0 = -\frac{4}{3}\sqrt{3}$ and $x_2 = +\frac{4}{3}\sqrt{3}$. Thus $\underline{\delta}_f(0) = -3.079201$ and $\overline{\delta}_f(0) = +3.079201$. Using these values we can plot the broken-line minorization of $f(x) = x^4 - 4x^2$ at $y = 0$. Compare this with the sharp quadratic minorization at $y = 0$, which is the function $g(x) = -4x^2$.



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