OPTIMIZING MONOTONICITY OF REGRESSION FUNCTIONS WITH THE PACKAGE `ORDINALS`

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ABSTRACT. Meet the abstract. This is the abstract.

1. INTRODUCTION

In ordinal `lineals` we optimize

$$\sigma(y_1,\cdots,y_m)=\sum_{j\neq\ell}^{m}\sum\{y_j'C_{j\ell}D_\ell^{-1}C_{\ell j}y_j-(y_j'C_{jl}y_\ell)^2\}$$

over increasing vectors of scores y_j, that are normalized by $u'D_jy_j=0$ and $y_j'D_jy_j=1$. Here the C_{jl} is the cross table for variables j and ℓ and D_j is the diagonal matrix of univariate marginals. Thus we minimize the difference over isotonic transformations of the correlation ratios and the squares of the product moment coefficients. In other words we try to linearize the regressions, and find a solution to the system of equations $\mathbf{E}(y_\ell|y_j)=\rho_{j\ell}y_j$ for all $j\neq\ell$.

Alternatively we may want to transform the variables such that the regression functions are increasing and not necessarily linear. This is, of course, a much more general requirement. In order to quantify this notion in a least squares loss function we define $m(m-1)$ additional vectors of unknowns, the *regression targets* $z_{j|\ell}$. Now an appropriate loss function is

$$\sigma(y_1,\cdots,y_m,z_{1|2},\cdots,z_{m-1|m})=\sum_{j\neq\ell}^{m}\sum(z_{j|\ell}-D_j^{-1}C_{j\ell}y_\ell)'D_j(z_{j|\ell}-D_j^{-1}C_{j\ell}y_\ell)$$

which should be minimized over all increasing y_j and over all increasing regression targets $z_{j|\ell}$, with the same normalization constraints on the y_j as before. This aims a find a solution to the system of inequalities that $\mathbf{E}(y_\ell|y_j)$ is increasing for each $j\neq\ell$.

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Finding the $z_{j\ell}$ for given y_j is just monotone regression. Finding the optimal y_j for given $z_{j|\ell}$ is slightly more complicated. Let

$$W_j = \sum_{\ell \neq j} C_{j\ell} D_\ell^{-1} C_{\ell j}.$$

Note that W_j is constant over iterations.

$$u_j = \sum_{\ell \neq j} C_{j\ell} z_{\ell|j}$$

and $v_j = W_j^{-1} u_j$. Then

$$\sigma(y_1, \cdots, y_m) = \sum_{j=1}^m (y_j - v_j)' W_j (y_j - v_j) - \sum_{j=1}^m v_j' W_j v_j + \sum_{j \neq \ell} \sum z_{j|\ell}' D_j z_{j|\ell}.$$

The variables separate, so we can solve for each y_j separately.

A quick way to improve y_j would be to use majorization. Suppose λ_j is the largest eigenvalue of $D_j^{-1} W_j$. The λ_j do not change during iteration and can be computed once and for all. Suppose $\tilde{y}_j$ is a candidate solution, for instance the current solution we are trying to improve.

$$\begin{aligned}
(y_j - v_j)' W_j (y_j - v_j) &= (\tilde{y}_j + (y_j - \tilde{y}_j) - v_j)' W_j (\tilde{y}_j + (y_j - \tilde{y}_j) - v_j) \\
&= (\tilde{y}_j - v_j)' W_j (\tilde{y}_j - v_j) + (y_j - \tilde{y}_j)' W_j (y_j - \tilde{y}_j) + 2(y_j - \tilde{y}_j)' W_j (\tilde{y}_j - v_j) \\
&\leq (\tilde{y}_j - v_j)' W_j (\tilde{y}_j - v_j) + \lambda_j (y_j - \tilde{y}_j)' D_j (y_j - \tilde{y}_j) + 2(y_j - \tilde{y}_j)' W_j (\tilde{y}_j - v_j).
\end{aligned}$$

Let

$$\hat{y}_j = \tilde{y}_j - \frac{1}{\lambda_j} D_j^{-1} W_j (\tilde{y}_j - v_j) = \tilde{y}_j - \frac{1}{\lambda_j} (D_j^{-1} W_j \tilde{y}_j - \tilde{u}_j),$$

with $\tilde{u}_j = D_j^{-1} u_j$. We see that a majorization step amounts to minimizing the simpler sum of squares $(y_j - \hat{y}_j)' D_j (y_j - \hat{y}_j)$ over increasing y_j satisfying $y_j' D_j y_j = 1$, and this is just monotone regression with normalization afterwards. The algorithm does not require computation of the inverses W_j^{-1} or the vectors v_j. We merely alternate a sequence of monotone regressions to update the targets $z_{j|\ell}$ and a sequence of monotone regressions to update the y_j.

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