NESTED AVERAGES

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Suppose $\mathcal{L}_1 \geq \cdots \geq \mathcal{L}_m$ are nested subspaces of an inner product space $\mathcal{L}$, and $P_1, \cdots, P_m$ are the corresponding orthogonal projectors.

For example, $\mathcal{L}$ are hourly time series of some fixed length, P_1 replaces the observations by daily averages, P_2 by weekly averages, P_3 by monthly averages, and so on.

Nestedness implies that the products $P(\alpha, \beta, \cdots, \mu) \stackrel{\Delta}{=} P_\alpha P_\beta \cdots P_\mu$ are equal to $P_{\max(\alpha,\beta,\cdots,\mu)}$. In particular it implies all $P(\alpha, \beta, \cdots, \mu)$ are symmetric and are, in fact, orthogonal projectors. The proof is simple. For all x we have $P_\ell x \in \mathcal{L}_\ell$. Thus $P_\ell x \in \mathcal{L}_k$ for all $k < \ell$, and $P_k P_\ell x = P_\ell x$. Since this is true for all x we have $P_k P_\ell = P_\ell$. Complete the proof by induction.

In our example daily averages of a series of monthly averages are obviously the monthly averages again. Thus $P_1 P_3 = P_3$.

If we define $y_k = P_k x$ then we can write

$$x - y_m = (y_{m-1} - y_m) + \cdots + (y_1 - y_2) + (x - y_1).$$

The matrices $P_{k-1} - P_k$ are orthogonal projectors on mutually orthogonal subspaces. Proof: suppose without loss of generality that $k > \ell$. Then also $k > \ell - 1$, $k - 1 > \ell - 1$, and $k - 1 \geq \ell$. Thus $(P_{k-1} - P_k)(P_{\ell-1} - P_\ell) = 0$. It follows that

$$\|x - y_m\|^2 = \|y_{m-1} - y_m\|^2 + \cdots + \|y_1 - y_2\|^2 + \|x - y_1\|^2.$$

In our example the

sum of squares of the data around monthly averages

equals the

sum of squares of the weekly averages around the monthly averages

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plus the

sum of squares of the daily averages around the weekly averages

plus the

sum of squares of the data around the daily averages.

This derivation may appear to be for finite vectors. But the same reasoning applies to random variables (with finite variance) on a given probability space, in which case the projectors are conditional expectations.

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