NON-MAJORIZATION OF PIECEWISE LINEAR FUNCTIONS

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ABSTRACT. A function $g : \mathbb{R}^n \Rightarrow \mathbb{R}$ majorizes a function $f : \mathbb{R}^n \Rightarrow \mathbb{R}$ at a point $y \in \mathbb{R}^n$ if $g(y) = f(y)$ and $g(x) \geq f(x)$ for all $x \in \mathbb{R}^n$. In optimization we are especially interested in quadratic majorizers [De Leeuw and Lange, 2009]. It is of some importance that the absolute value function has no quadratic majorizer at the origin. In this note we generalize this result.

1. RESULT

Suppose $f(x) = \max_{i=1}^n p_i'x + q_i$, where all p_i are different. Define the index set $I(x) = \{i \mid f(x) = p_i x + q_i\}$.

Theorem 1. *If $I(y)$ has two or more elements, then there is no quadratic majorizer of f at y.*

Proof. Define $g(x) = \frac{1}{2}x'Ax + b'x + c$ and $h(x) = g(x) - f(x)$. Then g is a majorizer of f at y if and only if h has a global minimum at y equal to zero. This implies that h has a local minimum at y, which implies that the directional derivative $h'(y,d)$ of h in y is non-negative in all directions d. But

$$h'(y,d) = \min_{i \in I(y)} d'(Ax + (b - p_i)),$$

and $h'(y,d) \geq 0$ if and only if $d'(Ax + (b - p_i) \geq 0$ for all d and for all $i \in I(y)$. If $i \neq k$ are in $I(y)$ then we must have both $Ax + (b - p_i) = 0$ and

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$Ax+(b-p_k)=0$, which implies $p_i=p_k$, contrary to our assumption that all p_i are different. □

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